Goldstein Chapter 1 Derivations

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1 Derivations

1. Show that for a single particle with constant mass the equation of motion implies the follwing differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

Answer:

$$\frac{dT}{dt} = \frac{d(\frac{1}{2}mv^2)}{dt} = m\mathbf{v} \cdot \dot{\mathbf{v}} = m\mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v}$$

with time variable mass,

$$\frac{d(mT)}{dt} = \frac{d}{dt}(\frac{p^2}{2}) = \mathbf{p} \cdot \dot{\mathbf{p}} = \mathbf{F} \cdot \mathbf{p}.$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation:

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2.$$

Answer:

$$M\mathbf{R} = \sum m_i \mathbf{r}_i$$

$$M^2\mathbf{R}^2 = \sum_{i,j} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j$$

Solving for $\mathbf{r}_i \cdot \mathbf{r}_j$ realize that $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. Square $\mathbf{r}_i - \mathbf{r}_j$ and you get

$$r_{ij}^2 = r_i^2 - 2\mathbf{r}_i \cdot \mathbf{r}_j + r_j^2$$

Plug in for $\mathbf{r}_i \cdot \mathbf{r}_j$

$$\mathbf{r}_i \cdot \mathbf{r}_j = \frac{1}{2}(r_i^2 + r_j^2 - r_{ij}^2)$$

$$M^2R^2 = \frac{1}{2}\sum_{i,j} m_i m_j r_i^2 + \frac{1}{2}\sum_{i,j} m_i m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = \frac{1}{2}M\sum_{i} m_i r_i^2 + \frac{1}{2}M\sum_{j} m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = M\sum_{i} m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

3. Suppose a system of two particles is known to obey the equations of motions,

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}$$

From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to the equations above.

Answer:

First, if the particles satisfy the strong law of action and reaction then they will automatically satisfy the weak law. The weak law demands that only the forces be equal and opposite. The strong law demands they be equal and opposite and lie along the line joining the particles. The first equation of motion tells us that internal forces have no effect. The equations governing the individual particles are

$$\dot{\mathbf{p}}_1 = \mathbf{F}_1^{(e)} + \mathbf{F}_{21}$$

$$\dot{\mathbf{p}}_2 = \mathbf{F}_2^{(e)} + \mathbf{F}_{12}$$

Assuming the equation of motion to be true, then

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \mathbf{F}_1^{(e)} + \mathbf{F}_{21} + \mathbf{F}_2^{(e)} + \mathbf{F}_{12}$$

must give

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

Thus $F_{12} = -F_{21}$ and they are equal and opposite and satisfy the weak law of action and reaction. If the particles obey

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}$$

then the time rate of change of the total angular momentum is only equal to the total external torque; that is, the internal torque contribution is null. For two particles, the internal torque contribution is

$$\mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12} = \mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times (-\mathbf{F}_{21}) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{21} = \mathbf{r}_{12} \times \mathbf{F}_{21} = 0$$

Now the only way for $\mathbf{r}_{12} \times \mathbf{F}_{21}$ to equal zero is for both $\mathbf{r}_{12}$ and $\mathbf{F}_{21}$ to lie on the line joining the two particles, so that the angle between them is zero, ie the magnitude of their cross product is zero.

$$\mathbf{A} \times \mathbf{B} = ABsin\theta$$

4. The equations of constraint for the rolling disk,

$$dx - a \sin\theta d\psi = 0$$

$$dy + a \cos\theta d\psi = 0$$

are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_n) dx_i = 0.$$

A constraint condition of this type is holonomic only if an integrating function $f(x_1, \ldots, x_n)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of the equations of constraint for the rolling disk.

Answer:

First attempt to find the integrating factor for the first equation. Note it is in the form:

$$Pdx + Qd\phi + Wd\theta = 0$$

where P is 1, Q is $-a\sin\theta$ and W is 0. The equations that are equivalent to

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

are

$$\frac{\partial(fP)}{\partial\phi} = \frac{\partial(fQ)}{\partial x}$$
$$\frac{\partial(fP)}{\partial\theta} = \frac{\partial(fW)}{\partial x}$$
$$\frac{\partial(fQ)}{\partial\theta} = \frac{\partial(fW)}{\partial\phi}$$

These are explicitly:

$$\frac{\partial(f)}{\partial\phi} = \frac{\partial(-fa\sin\theta)}{\partial x}$$
$$\frac{\partial(f)}{\partial\theta} = 0$$
$$\frac{\partial(-fa\sin\theta)}{\partial\theta} = 0$$

Simplfying the last two equations yields:

$$f\cos\theta = 0$$

Since y is not even in this first equation, the integrating factor does not depend on y and because of $\frac{\partial f}{\partial\theta} = 0$ it does not depend on θ either. Thus

$$f = f(x, \phi)$$

The only way for f to satisfy this equation is if f is constant and thus apparently there is no integrating function to make these equations exact. Performing the same procedure on the second equation you can find

$$\frac{\partial(fa\cos\theta)}{\partial y} = \frac{\partial f}{\partial\phi}$$

$$a\cos\theta\frac{\partial f}{\partial y} = \frac{\partial f}{\partial\phi}$$

and

$$f\sin\theta = 0$$

$$\frac{\partial f}{\partial \theta} = 0$$

leading to

$$f = f(y, \phi)$$

and making it impossible for f to satsify the equations unless as a constant. If this question was confusing to you, it was confusing to me too. Mary Boas says it is 'not usually worth while to spend much time searching for an integrating factor' anyways. That makes me feel better.

5. Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a palne. Show that there are two nonholonomic equations of constraint,

$$\cos\theta dx + \sin\theta dy = 0$$

$$\sin\theta dx - \cos\theta dy = \frac{1}{2}a(d\phi + d\phi')$$

(where θ,ϕ, and ϕ' have meanings similar to those in the problem of a single vertical disk, and (x,y) are the corrdinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

where C is a constant.

Answer:

The trick to this problem is carefully looking at the angles and getting the signs right. I think the fastest way to solve this is to follow the same procedure that was used for the single disk in the book, that is, find the speed of the disk, find the point of contact, and take the derivative of the x component, and y component of position, and solve for the equations of motion. Here the steps are taken a bit further because a holonomic relationship can be found that relates θ, ϕ and ϕ'. Once you have the equations of motion, from there its just slightly tricky algebra. Here goes:

We have two speeds, one for each disk

$$v' = a\dot{\phi}'$$

$$v = a\dot{\phi}$$

and two contact points,

$$(x \pm \frac{b}{2}\cos\theta, y \pm \frac{b}{2}\sin\theta)$$

The contact points come from the length of the axis being b as well as x and y being the center of the axis. The components of the distance are cos and sin for x and y repectively.

So now that we've found the speeds, and the points of contact, we want to take the derivatives of the x and y parts of their contact positions. This will give us the components of the velocity. Make sure you get the angles right, they were tricky for me.

$$\frac{d}{dt}(x + \frac{b}{2}\cos\theta) = v_x$$

$$\dot{x} - \frac{b}{2}\sin\theta\dot{\theta} = v\cos(180 - \theta - 90) = v\cos(90 - \theta) = v\cos(-90 + \theta) = v\sin\theta$$

$$\dot{x} - \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}\sin\theta$$

Do this for the next one, and get:

$$\dot{x} + \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}'\sin\theta$$

The plus sign is there because of the derivative of cos multiplied with the negative for the primed wheel distance from the center of the axis. For the y parts:

$$\frac{d}{dt}(y + \frac{b}{2}\sin\theta) = v_y$$

$$\dot{y} + \frac{b}{2}\cos\theta\dot{\theta} = -v\cos\theta = -a\dot{\phi}\cos\theta$$

It is negative because I decided to have axis in the first quadrent heading south-east. I also have the primed wheel south-west of the non-primed wheel. A picture would help, but I can't do that on latex yet. So just think about it.

Do it for the next one and get:

$$\dot{y} - \frac{b}{2}\cos\theta\dot{\theta} = -a\dot{\phi}'\cos\theta$$

All of the derivatives together so you aren't confused what I just did:

$$\dot{x} - \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}\sin\theta$$

$$\dot{x} + \frac{b}{2}\sin\theta\dot{\theta} = a\dot{\phi}'\sin\theta$$

$$\dot{y} + \frac{b}{2}\cos\theta\dot{\theta} = -a\dot{\phi}\cos\theta$$

$$\dot{y} - \frac{b}{2}\cos\theta\dot{\theta} = -a\dot{\phi}'\cos\theta$$

Now simplify them by cancelling the $dt's$ and leaving the x and y's on one side:

$$dx = \sin\theta[\frac{b}{2}d\theta + ad\phi] \tag{1}$$

$$dx = \sin\theta[-\frac{b}{2}d\theta + ad\phi'] \tag{2}$$

$$dy = -\cos\theta[\frac{b}{2}d\theta + ad\phi] \tag{3}$$

$$dy = -\cos\theta[-\frac{b}{2}d\theta + ad\phi'] \tag{4}$$

Now we are done with the physics. The rest is manipulation of these equations of motion to come up with the constraints. For the holonomic equation use (1)-(2).

$$(1) - (2) = 0 = bd\theta + a(d\phi - d\phi')$$

$$d\theta = -\frac{a}{b}(d\phi - d\phi')$$

$$\theta = -\frac{a}{b}(\phi - \phi') + C$$

For the other two equations, I started with

$$(1)\cos\theta + (3)\sin\theta = \cos\theta\sin\theta[\frac{b}{2}d\theta + ad\phi] - \sin\theta\cos\theta[\frac{b}{2}d\theta + ad\phi]$$

$$\cos\theta dx + \sin\theta dy = 0$$

and

$$(1) + (2) = 2dx = \sin\theta a[d\phi + d\phi']$$

$$(3) + (4) = 2dy = -\cos\theta a[d\phi + d\phi']$$

multiply dy by $-\cos\theta$ and multiply dx by $\sin\theta$ to yield yourself

$$-\cos\theta dy = \cos^2\theta\frac{a}{2}[d\phi + d\phi']$$

$$\sin\theta dx = \sin^2\theta\frac{a}{2}[d\phi + d\phi']$$

Add them together and presto!

$$\sin\theta dx - \cos\theta dy = \frac{a}{2}[d\phi + d\phi']$$

6. A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, *the constraint is nonholonomic.*

Answer:

The abscissa is the x-axis distance from the origin to the point on the x-axis that the velocity vector is aimed at. It has the distance $f(t)$.

I claim that the ratio of the velocity vector components must be equal to the ratio of the vector components of the vector that connects the particle to the point on the x-axis. The directions are the same. The velocity vector components are:

$$v_y = \frac{dy}{dt}$$

$$v_x = \frac{dx}{dt}$$

The vector components of the vector that connects the particle to the point on the x-axis are:

$$V_y = y(t)$$

$$V_x = x(t) - f(t)$$

For these to be the same, then

$$\frac{v_y}{v_x} = \frac{V_y}{V_x}$$

$$\frac{dy}{dx} = \frac{y(t)}{x(t) - f(t)}$$

$$\frac{dy}{y(t)} = \frac{dx}{x(t) - f(t)}$$

This cannot be integrated with $f(t)$ being arbituary. Thus the constraint is nonholonomic. It's nice to write the constraint in this way because it's frequently the type of setup Goldstein has:

$$ydx + (f(t) - x)dy = 0$$

There can be no integrating factor for this equation.

7. The Lagrangian equations can be written in the form of the Nielsen's equations.

$$\frac{\partial \dot{T}}{\partial \dot{q}} - 2\frac{\partial T}{\partial q} = Q$$

Show this.

Answer:

I'm going to set the two forms equal and see if they match. That will show that they can be written as displayed above.

$$\text{Lagrangian Form} = \text{Nielsen's Form}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) - \frac{\partial T}{\partial q} = \frac{\partial \dot{T}}{\partial \dot{q}} - 2\frac{\partial T}{\partial q}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) + \frac{\partial T}{\partial q} = \frac{\partial \dot{T}}{\partial \dot{q}} \tag{5}$$

What is $\frac{\partial \dot{T}}{\partial \dot{q}}$ you may ask? Well, lets solve for $\dot{T}$ first.

$$\dot{T} \equiv \frac{d}{dt}T(q, \dot{q}, t)$$

Because $\frac{d}{dt}$ is a full derivative, you must not forget the chain rule.

$$\dot{T} \equiv \frac{d}{dt}T(q, \dot{q}, t) = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q}$$

Now lets solve for $\frac{\partial \dot{T}}{\partial \dot{q}}$, not forgetting the product rule

$$\frac{\partial \dot{T}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}[\frac{\partial T}{\partial t} + \frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q}]$$

$$\frac{\partial \dot{T}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\frac{\partial T}{\partial t} + \frac{\partial}{\partial \dot{q}}\frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial q}\frac{\partial \dot{q}}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\frac{\partial T}{\partial \dot{q}}\ddot{q}$$

$$\frac{\partial \dot{T}}{\partial \dot{q}} = \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q} + \frac{\partial T}{\partial q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

Now we have $\frac{\partial \dot{T}}{\partial \dot{q}}$, so lets plug this into equation (5).

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) + \frac{\partial T}{\partial q} = \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q} + \frac{\partial T}{\partial q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) = \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

Notice that this is indeed true.

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}}) = \frac{\partial}{\partial t}(\frac{\partial T}{\partial \dot{q}}) + \frac{\partial}{\partial q}(\frac{\partial T}{\partial \dot{q}})\dot{q} + \frac{\partial}{\partial \dot{q}}(\frac{\partial T}{\partial \dot{q}})\ddot{q}$$

because $T = T(q, \dot{q}, t)$.

If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, ..., q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

Answer:

Let's directly substitute L' into Lagrange's equations.

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} - \frac{\partial L'}{\partial q} = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}(L + \frac{dF}{dt}) - \frac{\partial}{\partial q}(L + \frac{dF}{dt}) = 0$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\frac{dF}{dt}] - \frac{\partial L}{\partial q} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

On the left we recognized Lagrange's equations, which we know equal zero. Now to show the terms with F vanish.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial \dot{F}}{\partial q}$$

This is shown to be true because

$$\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

We have

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial F}{\partial q}$$

$$= \frac{\partial}{\partial t}\frac{\partial F}{\partial q} + \frac{\partial}{\partial q}\frac{\partial F}{\partial q}\dot{q}$$

$$= \frac{\partial}{\partial q}[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}] = \frac{\partial \dot{F}}{\partial q}$$

Thus as Goldstein reminded us, $L = T - V$ is a suitable Lagrangian, but it is not the only Lagrangian for a given system.

9. The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \to \mathbf{A} + \nabla\psi(\mathbf{r}, \mathbf{t})$$

$$\phi \to \phi - \frac{1}{c}\frac{\partial\psi}{\partial t}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

Answer:

$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\mathbf{A} \cdot \mathbf{v}$$

Upon the gauge transformation:

$$L' = \frac{1}{2}mv^2 - q[\phi - \frac{1}{c}\frac{\partial\psi}{\partial t}] + \frac{q}{c}[\mathbf{A} + \nabla\psi(r,t)] \cdot \mathbf{v}$$

$$L' = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\mathbf{A} \cdot \mathbf{v} + \frac{q}{c}\frac{\partial\psi}{\partial t} + \frac{q}{c}\nabla\psi(r,t) \cdot \mathbf{v}$$

$$L' = L + \frac{q}{c}[\frac{\partial\psi}{\partial t} + \nabla\psi(r,t) \cdot \mathbf{v}]$$

$$L' = L + \frac{q}{c}[\dot{\psi}]$$

In the previous problem it was shown that:

$$\frac{d}{dt}\frac{\partial\dot{\psi}}{\partial\dot{q}} = \frac{\partial\dot{\psi}}{\partial q}$$

For ψ differentiable but arbitrary. This is all that you need to show that the Lagrangian is changed but the motion is not. This problem is now in the same form as before:

$$L' = L + \frac{dF(q_1, ..., q_n, t)}{dt}$$

And if you understood the previous problem, you'll know why there is no effect on the motion of the particle(i.e. there are many Lagrangians that may describe the motion of a system, there is no unique Lagrangian).

10. Let $q_1, ..., q_n$ be a set of independent generalized coordinates for a system *of n degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, ..., s_n$ by means of transformation equations*

$$q_i = q_i(s_1, ..., s_n, t), \qquad i = 1, ..., n.$$

(Such a transformatin is called a point transformation.) Show that if the Lagrangian function is expressed as a function of $s_j, \dot{s}_j$ and t through the equation of transformation, then L satisfies Lagrange's equations with respect to the s coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = 0$$

In other words, the form of Lagrange's equations is invariant under a point transformation.

Answer:

We know:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

and we want to prove:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = 0$$

If we put $\frac{\partial L}{\partial \dot{s}_j}$ and $\frac{\partial L}{\partial s_j}$ in terms of the q coordinates, then they can be substitued back in and shown to still satisfy Lagrange's equations.

$$\frac{\partial L}{\partial s_j} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j}$$

$$\frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{s}_j}$$

We know:

$$\frac{\partial q_i}{\partial s_j} = \frac{\partial \dot{q}_i}{\partial \dot{s}_j}$$

Thus,

$$\frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial s_j}$$

Plug $\frac{\partial L}{\partial \dot{s}_j}$ and $\frac{\partial L}{\partial s_j}$ into the Lagrangian equation and see if they satisfy it:

$$\frac{d}{dt}[\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial s_j}] - [\sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j}] = 0$$

Pulling out the summation to the right and $\frac{\partial q_i}{\partial s_j}$ to the left, we are left with:

$$\sum_i [\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}]\frac{\partial q_i}{\partial s_j} = 0$$

This shows that Lagrangian's equations are invariant under a point transformation.