Goldstein Chapter 1 Exercises

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1 Exercises

11. Consider a uniform thin disk that rolls without slipping on a horizontal plane. A horizontal force is applied to the center of the disk and in a direction parallel to the plane of the disk.

- Derive Lagrange's equations and find the generalized force.
- Discuss the motion if the force is not applied parallel to the plane of the disk.

Answer:

To find Lagrangian's equations, we need to first find the Lagrangian.

$$L = T - V$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2 \qquad V = 0$$

Therefore

$$L = \frac{1}{2}m(r\omega)^2$$

Plug into the Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q$$

$$\frac{d}{dt}\frac{\partial \frac{1}{2}mr^2\omega^2}{\partial (r\omega)} - \frac{\partial \frac{1}{2}mr^2\omega^2}{\partial x} = Q$$

$$\frac{d}{dt}m(r\omega) = Q$$

$$m(r\ddot{\omega}) = Q$$

If the motion is not applied parallel to the plane of the disk, then there might be some slipping, or another generalized coordinate would have to be introduced, such as θ to describe the y-axis motion. The velocity of the disk would not just be in the x-direction as it is here.

12. The escape velocity of a particle on Earth is the minimum velocity required at Earth's surface in order that that particle can escape from Earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorme for potential plus kinetic energy show that the escape veolcity for Earth, ingnoring the presence of the Moon, is 11.2 km/s.

Answer:

$$\frac{GMm}{r} = \frac{1}{2}mv^2$$
$$\frac{GM}{r} = \frac{1}{2}v^2$$

Lets plug in the numbers to this simple problem:

$$\frac{(6.67 \times 10^{-11}) \cdot (6 \times 10^{24})}{(6 \times 10^6)} = \frac{1}{2}v^2$$

This gives $v = 1.118 \times 10^4$ m/s which is 11.2 km/s.

13. Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the raction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is:

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initally from rest, with v' equal to 2.1 km/s and a mass loss per second equal to 1/60th of the intial mass, that in order to reach the escape velocity the ratio of the wight of the fuel to the weight of the empty rocket must be almost 300!

Answer:

This problem can be tricky if you're not very careful with the notation. But here is the best way to do it. Defining m_e equal to the empty rocket mass, m_f is the total fuel mass, m_0 is the intitial rocket mass, that is, $m_e + m_f$, and $\frac{dm}{dt} = -\frac{m_0}{60}$ as the loss rate of mass, and finally the goal is to find the ratio of

m_f/m_e to be about 300.

The total force is just ma, as in Newton's second law. The total force on the rocket will be equal to the force due to the gas escaping minus the weight of the rocket:

$$ma = \frac{d}{dt}[-mv'] - mg$$

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg$$

The rate of lost mass is negative. The velocity is in the negative direction, so, with the two negative signs the term becomes positive.

Use this:

$$\frac{dv}{dm}\frac{dm}{dt} = \frac{dv}{dt}$$

Solve:

$$m\frac{dv}{dm}\frac{dm}{dt} = -v'\frac{dm}{dt} - mg$$

$$\frac{dv}{dm}\frac{dm}{dt} = -\frac{v'}{m}\frac{dm}{dt} - g$$

$$\frac{dv}{dm} = -\frac{v'}{m} + \frac{60g}{m_0}$$

Notice that the two negative signs cancelled out to give us a positive far right term.

$$dv = -\frac{v'}{m}dm + \frac{60g}{m_0}dm$$

Integrating,

$$\int dv = -v'\int_{m_0}^{m_e}\frac{dm}{m} + \int_{m_0}^{m_e}\frac{60g}{m_0}dm$$

$$v = -v'\ln\frac{m_e}{m_0} + \frac{60g}{m_0}(m_e - m_0)$$

$$v = -v'\ln\frac{m_e}{m_e + m_f} + 60g\frac{m_e - m_e - m_f}{m_e + m_f}$$

$$v = v'\ln\frac{m_e + m_f}{m_e} - 60g\frac{m_f}{m_e + m_f}$$

Now watch this, I'm going to use my magic wand of approximation. This is when I say that because I know that the ratio is so big, I can ignore the empty rocket mass as compared to the fuel mass. $m_e << m_f$. Let me remind you, we are looking for this ratio as well. The ratio of the fuel mass to empty rocket, m_f/m_e.

$$v = v' \ln \frac{m_e + m_f}{m_e} - 60g \frac{m_f}{m_e + m_f}$$

$$v = v' \ln \frac{m_f}{m_e} - 60g \frac{m_f}{m_f}$$

$$\frac{v + 60g}{v'} = \ln \frac{m_f}{m_e}$$

$$\exp[\frac{v + 60g}{v'}] = \frac{m_f}{m_e}$$

Plug in 11,200 m/s for v, 9.8 for g, and 2100 m/s for v'.

$$\frac{m_f}{m_e} = 274$$

And, by the way, if Goldstein hadn't just converted 6800 ft/s from his second edition to 2.1 km/s in his third edition without checking his answer, he would have noticed that 2.07 km/s which is a more accurate approximation, yields a ratio of 296. This is more like the number 300 he was looking for.

14. Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

Answer:

$$T_1 + T_2 = T$$

Where T_1 equals the kinetic energy of the center of mass, and T_2 is the kinetic energy about the center of mass. Keep these two parts seperate!

Solve for T_1 first, its the easiest:

$$T_1 = \frac{1}{2} M v_{cm}^2 = \frac{1}{2}(2m)(a\dot{\psi})^2 = ma^2\dot{\psi}^2$$

Solve for T_2, realizing that the rigid rod is not restricted to just the X-Y plane. Don't forget the Z-axis!

$$T_2 = \frac{1}{2} M v^2 = mv^2$$

Solve for v^2 about the center of mass. The angle ϕ will be the angle in the x-y plane, while the angle θ will be the angle from the z-axis.

If $\theta = 90^o$ and $\phi = 0^o$ then $x = l/2$ so:

$$x = \frac{l}{2}\sin\theta\cos\phi$$

If $\theta = 90^o$ and $\phi = 90^o$ then $y = l/2$ so:

$$y = \frac{l}{2}\sin\theta\sin\phi$$

If $\theta = 0^o$, then $z = l/2$ so:

$$z = \frac{l}{2}\cos\theta$$

Find v^2:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$$

$$\dot{x} = \frac{l}{2}(\cos\phi\cos\theta\dot{\theta} - \sin\theta\sin\phi\dot{\phi})$$

$$\dot{y} = \frac{1}{2}(\sin\phi\cos\theta\dot{\theta} + \sin\theta\cos\phi\dot{\phi})$$

$$\dot{z} = -\frac{l}{2}\sin\theta\dot{\theta}$$

Carefully square each:

$$\dot{x}^2 = \frac{l^2}{4}\cos^2\phi\cos^2\theta\dot{\theta}^2 - 2\frac{l}{2}\sin\theta\sin\phi\dot{\phi}\frac{l}{2}\cos\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\sin^2\phi\dot{\phi}^2$$

$$\dot{y}^2 = \frac{l^2}{4}\sin^2\phi\cos^2\theta\dot{\theta}^2 + 2\frac{l}{2}\sin\theta\cos\phi\dot{\phi}\frac{l}{2}\sin\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\cos^2\phi\dot{\phi}^2$$

$$\dot{z}^2 = \frac{l^2}{4}\sin^2\theta\dot{\theta}^2$$

Now add, striking out the middle terms:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \frac{l^2}{4}[\cos^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\sin^2\phi\dot{\phi}^2+\sin^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\cos^2\phi\dot{\phi}^2+\sin^2\theta\dot{\theta}^2]$$

Pull the first and third terms inside the brackets together, and pull the second and fourth terms together as well:

$$v^2 = \frac{l^2}{4}[\cos^2\theta\dot{\theta}^2(\cos^2\phi + \sin^2\phi) + \sin^2\theta\dot{\phi}^2(\sin^2\phi + \cos^2\phi) + \sin^2\theta\dot{\theta}^2]$$

$$v^2 = \frac{l^2}{4}(\cos^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

$$v^2 = \frac{l^2}{4}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

Now that we finally have v^2 we can plug this into T_2

$$T = T_1 + T_2 = ma^2\dot{\psi}^2 + m\frac{l^2}{4}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

It was important to emphasize that T_1 is the kinetic energy of the total mass around the center of the circle while T_2 is the kinetic energy of the masses about the center of mass. Hope that helped.

15. A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \sigma \cdot \mathbf{L}$$

where r is the radius vector from a fixed point, L is the angular momentum about that point, and σ is a fixed vector in space.

1. Find the components of the force on the particle in both Cartesian and spherical poloar coordinates, on the basis of Lagrangian's equations with a generalized potential
2. Show that the components in the two coordinate systems are related to each other as in the equation shown below of generalized force
3. Obtain the equations of motion in spherical polar coordinates

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Answer:

This one is a fairly tedious problem mathematically. First lets find the components of the force in Cartesian coordinates. Convert $U(r, v)$ into Cartesian and then plug the expression into the Lagrange-Euler equation.

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma \cdot (r \times p)] - \frac{\partial}{\partial q_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma \cdot (r \times p)]$$

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{v}_j}[\sigma \cdot [(x\hat{i} + y\hat{j} + z\hat{k}) \times (p_x\hat{i} + p_y\hat{j} + p_z\hat{k})] - \frac{\partial}{\partial x_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma \cdot (r \times p)]$$

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{v}_j}[\sigma \cdot [(yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - p_x y)\hat{k}] - \frac{\partial}{\partial x_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma \cdot (r \times p)]$$

$$Q_j = \frac{d}{dt}\frac{\partial}{\partial \dot{v}_j}[m\sigma_x(yv_z - zv_y) + m\sigma_y(zv_x - xv_z) + m\sigma_z(xv_y - v_x y)] - \frac{\partial}{\partial x_j}[V(\sqrt{x^2+y^2+z^2}) + \sigma \cdot (r \times p)]$$

Where we know that

$$m\sigma_x(yv_z - zv_y) + m\sigma_y(zv_x - xv_z) + m\sigma_z(xv_y - v_x y) = \sigma \cdot (r \times p)$$

So lets solve for just one component first and let the other ones follow by example:

$$Q_x = \frac{d}{dt}(m\sigma_y z - m\sigma_z y) - \frac{\partial}{\partial x}[V(\sqrt{x^2+y^2+z^2}) + m\sigma_x(yv_z - zv_y) + m\sigma_y(zv_x - xv_z) + m\sigma_z(xv_y - v_x y)]$$

$$Q_x = m(\sigma_y v_z - \sigma_z v_y) - [V'(\sqrt{x^2+y^2+z^2})(x^2+y^2+z^2)^{-\frac{1}{2}} x - m\sigma_y v_z + m\sigma_z v_y]$$

$$Q_x = 2m(\sigma_y v_z - \sigma_z v_y) - V'\frac{x}{r}$$

If you do the same for the y and z components, they are:

$$Q_y = 2m(\sigma_z v_x - \sigma_x v_z) - V'\frac{y}{r}$$

$$Q_z = 2m(\sigma_x v_y - \sigma_y v_x) - V'\frac{z}{r}$$

Thus the generalized force is:

$$F = 2m(\sigma \times \mathbf{v}) - V'\frac{\mathbf{r}}{r}$$

Now its time to play with spherical coordinates. The trick to this is setting up the coordinate system so that σ is along the z axis. Thus the dot product simplifies and L is only the z-component.

$$U = V(r) + m\sigma(x\dot{y} - y\dot{x})$$

With spherical coordinate definitions:

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

Solving for $(x\dot{y} - y\dot{x})$

$$\dot{x} = r(-\sin\theta\sin\phi\dot{\phi} + \cos\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\cos\phi$$
$$\dot{y} = r(\sin\theta\cos\phi\dot{\phi} + \sin\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\sin\phi$$

Thus $x\dot{y} - y\dot{x}$ is

$$= r\sin\theta\cos\phi[r(\sin\theta\cos\phi\dot{\phi} + \sin\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\sin\phi]$$
$$-r\sin\theta\sin\phi[r(-\sin\theta\sin\phi\dot{\phi} + \cos\phi\cos\theta\dot{\theta}) + \dot{r}\sin\theta\cos\phi].$$

Note that the $\dot{r}$ terms drop out as well as the $\dot{\theta}$ terms.

$$x\dot{y} - y\dot{x} = r^2\sin^2\theta\cos^2\phi\dot{\phi} + r^2\sin^2\theta\sin^2\phi\dot{\phi}$$
$$x\dot{y} - y\dot{x} = r^2\sin^2\theta\dot{\phi}$$

Thus

$$U = V(r) + m\sigma r^2\sin^2\theta\dot{\phi}$$

Plugging this in to Lagrangian's equations yields:

For Q_r:

$$Q_r = -\frac{\partial U}{\partial r} + \frac{d}{dt}(\frac{\partial U}{\partial \dot{r}})$$
$$Q_r = -\frac{dV}{dr} - 2m\sigma r\sin^2\theta\dot{\phi} + \frac{d}{dt}(0)$$
$$Q_r = -\frac{dV}{dr} - 2m\sigma r\sin^2\theta\dot{\phi}$$

For Q_θ:

$$Q_\theta = -2m\sigma r^2\sin\theta\dot{\phi}\cos\theta$$

For Q_ϕ:

$$Q_\phi = \frac{d}{dt}(m\sigma r^2\sin^2\theta)$$
$$Q_\phi = m\sigma(r^2 2\sin\theta\cos\theta\dot{\theta} + \sin^2\theta 2r\dot{r})$$
$$Q_\phi = 2m\sigma r^2\sin\theta\cos\theta\dot{\theta} + 2m\sigma r\dot{r}\sin^2\theta$$

For part b, we have to show the components of the two coordinate systems are related to each other via

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Lets take ϕ for an example,

$$Q_\phi = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \phi} = F_x\frac{\partial x}{\partial \phi} + F_y\frac{\partial y}{\partial \phi} + F_z\frac{\partial z}{\partial \phi}$$

$$Q_\phi = Q_x(-r\sin\theta\sin\phi) + Q_y(r\sin\theta\cos\phi) + Q_z(0)$$

$$Q_\phi = [2m(\sigma_y v_z - \sigma_z v_y) - V'\frac{x}{r}](-r\sin\theta\sin\phi) + [2m(\sigma_z v_x - \sigma_x v_z) - V'\frac{y}{r}](r\sin\theta\cos\phi) + 0$$

Because in both coordinate systems we will have σ pointing in only the z direction, then the x and y σ's disappear:

$$Q_\phi = [2m(-\sigma_z v_y) - V'\frac{x}{r}](-r\sin\theta\sin\phi) + [2m(\sigma_z v_x) - V'\frac{y}{r}](r\sin\theta\cos\phi)$$

Pull out the V' terms, plug in x and y, see how V' terms cancel

$$Q_\phi = V'(x\sin\theta\sin\phi - y\sin\theta\cos\phi) - 2mr\sin\theta\sigma[v_y\sin\phi + v_x\cos\phi]$$

$$Q_\phi = V'(r\sin^2\theta\cos\phi\sin\phi - r\sin^2\theta\sin\phi\cos\phi) - 2mr\sin\theta\sigma[v_y\sin\phi + v_x\cos\phi]$$

$$Q_\phi = -2mr\sin\theta\sigma[v_y\sin\phi + v_x\cos\phi]$$

Plug in v_y and v_x:

$$Q_\phi = -2mr\sin\theta\sigma[\sin\phi(r\sin\theta\cos\phi\dot{\phi} + r\sin\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\sin\phi)$$

$$+\cos\phi(-r\sin\theta\sin\phi\dot{\phi} + r\cos\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\cos\phi)].$$

$$Q_\phi = 2m\sigma r\sin\theta[r\sin^2\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\sin^2\phi$$
$$+r\cos^2\phi\cos\theta\dot{\theta} + \dot{r}\sin\theta\cos^2\phi].$$

Gather sin^2's and cos^2's:

$$Q_\phi = 2m\sigma r\sin\theta[r\cos\theta\dot{\theta} + \dot{r}\sin\theta]$$

This checks with the derivation in part a for Q_ϕ. This shows that indeed the components in the two coordinate systems are related to each other as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Any of the other components could be equally compared in the same procedure. I chose Q_ϕ because I felt it was easiest to write up.

For part c, to obtain the equations of motion, we need to find the generalized kinetic energy. From this we'll use Lagrange's equations to solve for each component of the force. With both derivations, the components derived from the generalized potential, and the components derived from kinetic energy, they will be set equal to each other.

In spherical coordinates, v is:

$$\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$$

The kinetic energy in spherical polar coordinates is then:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$$

For the r component:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{r}}) - \frac{\partial T}{\partial r} = Q_r$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 = Q_r$$

$$m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 = Q_r$$

From part a,

$$Q_r = -V' - 2m\sigma r\sin^2\theta\dot{\phi}$$

Set them equal:

$$m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 = Q_r = -V' - 2m\sigma r\sin^2\theta\dot{\phi}$$

$$m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\dot{\phi}^2 + V' + 2m\sigma r\sin^2\theta\dot{\phi} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}(2\sigma - \dot{\phi}) + V' = 0$$

For the θ component:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}}) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\dot{\phi}^2\cos\theta = Q_\theta$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\dot{\phi}^2\cos\theta = Q_\theta$$

From part a,

$$Q_\theta = -2m\sigma r^2\sin\theta\cos\theta\dot{\phi}$$

Set the two equal:

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\dot{\phi}^2\cos\theta + 2m\sigma r^2\sin\theta\cos\theta\dot{\phi} = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mr^2\sin\theta\cos\theta\dot{\phi}(2\sigma - \dot{\phi}) = 0$$

For the last component, ϕ we have:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\phi}}) - \frac{\partial T}{\partial \phi} = Q_\phi$$

$$\frac{d}{dt}(mr^2 \sin^2\theta\dot{\phi}) - 0 = Q_\phi$$

$$mr^2\frac{d}{dt}(\sin^2\theta\dot{\phi}) + 2mr\dot{r}\sin^2\theta\dot{\phi} = Q_\phi$$

$$mr^2\sin^2\theta\ddot{\phi} + 2mr^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 2mr\dot{r}\sin^2\theta\dot{\phi} = Q_\phi$$

From part a,

$$Q_\phi = 2m\sigma r^2\sin\theta\cos\theta\dot{\theta} + 2m\sigma r\dot{r}\sin^2\theta$$

Set the two equal:

$$mr^2\sin^2\theta\ddot{\phi}+2mr^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}+2mr\dot{r}\sin^2\theta\dot{\phi}-2m\sigma r^2\sin\theta\cos\theta\dot{\theta}-2m\sigma r\dot{r}\sin^2\theta = 0$$

$$mr^2\sin^2\theta\ddot{\phi} + 2mr^2\sin\theta\cos\theta\dot{\theta}(\dot{\phi}-\sigma) + 2mr\dot{r}\sin^2\theta(\dot{\phi}-\sigma) = 0$$

That's it, here are all of the equations of motion together in one place:

$$m\ddot{r} - mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}(2\sigma-\dot{\phi}) + V' = 0$$
$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mr^2\sin\theta\cos\theta\dot{\phi}(2\sigma-\dot{\phi}) = 0$$
$$mr^2\sin^2\theta\ddot{\phi} + 2mr^2\sin\theta\cos\theta\dot{\theta}(\dot{\phi}-\sigma) + 2mr\dot{r}\sin^2\theta(\dot{\phi}-\sigma) = 0$$

16. A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2})$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. The expression for F represents the force between two charges in Weber's electrodynamics.

Answer:

This one takes some guess work and careful handling of signs. To get from force to potential we will have to take a derivative of a likely potential. Note that if you expand the force it looks like this:

$$F = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r}$$

We know that

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = F$$

So lets focus on the time derivative for now. If we want a $\ddot{r}$ we would have to take the derivative of a $\dot{r}$. Let pick something that looks close, say $\frac{2\dot{r}}{c^2r}$:

$$\frac{d}{dt}(\frac{2\dot{r}}{c^2r}) = \frac{2\dot{r}}{c^2}(-\frac{\dot{r}}{r^2}) + \frac{2\ddot{r}}{c^2r} = -\frac{2\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r}$$

Excellent! This has our third term we were looking for. Make this stay the same when you take the partial with respect to $\dot{r}$.

$$\frac{\partial}{\partial \dot{r}}\frac{\dot{r}^2}{c^2r} = \frac{2\dot{r}}{c^2r}$$

So we know that the potential we are guessing at, has the term $\frac{\dot{r}^2}{c^2r}$ in it. Lets add to it what would make the first term of the force if you took the negative partial with respect to r, see if it works out.

That is,

$$-\frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r^2}$$

So

$$U = \frac{1}{r} + \frac{\dot{r}^2}{c^2r}$$

might work. Checking:

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = F$$

We have

$$\frac{\partial U}{\partial r} = -\frac{1}{r^2} - \frac{\dot{r}^2}{c^2r^2}$$

and

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{d}{dt}\frac{2\dot{r}}{c^2r} = \frac{2\dot{r}}{c^2}(-\frac{\dot{r}}{r^2}) + \frac{1}{r}(\frac{2\ddot{r}}{c^2}) = -\frac{2\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r}$$

thus

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{1}{r^2} + \frac{\dot{r}^2}{c^2r^2} - \frac{2\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r}$$

$$-\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r}$$

This is indeed the force unexpanded,

$$F = \frac{1}{r^2}(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}) = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r}$$

Thus our potential, $U = \frac{1}{r} + \frac{\dot{r}^2}{c^2r}$ works. To find the Lagrangian use $L = T - U$. In a plane, with spherical coordinates, the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Thus

$$L = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\theta}^2) - \frac{1}{r}(1 + \frac{\dot{r}^2}{c^2})$$

17. A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. The MeV, million electron volt, is a unit of energy used in modern physics equal to 1.60×10^{-13} J. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34×10^{-22} kg·m/s. In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90×10^{-25} kg what is its kinetic energy, in electron volts?

Answer:

If you draw a diagram you'll see that the nucleus recoils in the opposite direction of the vector made by the electron plus the neutrino emission. Place the neutrino at the x-axis, the electron on the y axis and use pythagorean's theorme to see the nucleus will recoil with a momentum of 2 Mev/c. The nucleus goes in the opposite direction of the vector that makes an angle

$$\theta = \tan^{-1}\frac{1.73}{1} = 60°$$

from the x axis. This is 240° from the x-axis.

To find the kinetic energy, you can convert the momentum to $kg \cdot m/s$, then convert the whole answer that is in joules to eV,

$$T = \frac{p^2}{2m} = \frac{[2(5.34 \times 10^{-22})]^2}{2 \cdot 3.9 \times 10^{-25}} \cdot \frac{1MeV}{1.6 \times 10^{-13}J} \cdot \frac{10^6eV}{1MeV} = 9.13eV$$

18. A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2).$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0$, $c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq. (1.57'):

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt}$$

is related to L' by a point transformation (cf. Derivation 10). What is the significance of the condition on the value of $b^2 - ac$?

Answer:

To find the equations of motion, use the Euler-Lagrange equations.

$$\frac{\partial L'}{\partial q} = \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}}$$

For x first:

$$-\frac{\partial L'}{\partial x} = -(-Kax - Kby) = K(ax + by)$$

$$\frac{\partial L'}{\partial \dot{x}} = m(a\dot{x} + b\dot{y})$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = m(a\ddot{x} + b\ddot{y})$$

Thus

$$-K(ax + by) = m(a\ddot{x} + b\ddot{y})$$

Now for y:

$$-\frac{\partial L'}{\partial y} = -(-Kby - Kcy) = K(bx + cy)$$

$$\frac{\partial L'}{\partial \dot{x}} = m(b\dot{x} + c\dot{y})$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = m(b\ddot{x} + c\ddot{y})$$

Thus

$$-K(bx + cy) = m(b\ddot{x} + c\ddot{y})$$

Therefore our equations of motion are:

$$-K(ax+by) = m(a\ddot{x}+b\ddot{y})$$
$$-K(bx+cy) = m(b\ddot{x}+c\ddot{y})$$

Examining the particular cases, we find:

If $a = 0 = c$ then:

$$-Kx = m\ddot{x} \quad -Ky = -m\ddot{y}$$

If $b = 0$, $c = -a$ then:

$$-Kx = m\ddot{x} \quad -Ky = -m\ddot{y}$$

The physical system is harmonic oscillation of a particle of mass m in two dimensions. If you make a substitution to go to a different coordinate system this is easier to see.

$$u = ax+by \quad v = bx+cy$$

Then

$$-Ku = m\ddot{u}$$
$$-Kv = m\ddot{v}$$

The system can now be more easily seen as two independent but identical simple harmonic oscillators, after a point transformation was made.

When the condition $b^2 - ac \neq 0$ is violated, then we have $b = \sqrt{ac}$, and L' simplifies to this:

$$L' = \frac{m}{2}(\sqrt{a}\dot{x}+\sqrt{c}\dot{y})^2 - \frac{K}{2}(\sqrt{a}x+\sqrt{c}y)^2$$

Note that this is now a one dimensional problem. So the condition keeps the Lagrangian in two dimensions, or you can say that the transformation matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is singluar because $b^2 - ac \neq 0$ Note that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

So if this condition holds then we can reduce the Lagrangian by a point transformation.

19. Obtain the Lagrange equations of motion for spherical pendulum, i.e., a mass point suspended by a rigid weightless rod.

Answer:

The kinetic energy is found the same way as in exercise 14, and the potential energy is found by using the origin to be at zero potential.

$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

If θ is the angle from the positive z-axis, then at $\theta = 90°$ the rod is aligned along the x-y plane, with zero potential. Because $cos(90) = 0$ we should expect a cos in the potential. When the rod is aligned along the z-axis, its potential will be its height.

$$V = mgl\cos\theta$$

If $\theta = 0$ then $V = mgl$. If $\theta = 180$ then $V = -mgl$.
So the Lagrangian is $L = T - V$.

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - mgl\cos\theta$$

To find the Lagrangian equations, they are the equations of motion:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$$

Solving these yields:

$$\frac{\partial L}{\partial \theta} = ml^2\sin\theta\dot{\phi}^2\cos\theta + mgl\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta}$$

Thus

$$ml^2\sin\theta\dot{\phi}^2\cos\theta + mgl\sin\theta - ml^2\ddot{\theta} = 0$$

and

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}(ml^2\sin^2\theta\dot{\phi}) = ml^2\sin^2\theta\ddot{\phi} + 2\dot{\phi}ml^2\sin\theta\cos\theta$$

Thus

$$ml^2\sin^2\theta\ddot{\phi} + 2\dot{\phi}ml^2\sin\theta\cos\theta = 0$$

Therefore the equations of motion are:

$$ml^2 \sin\theta\dot{\phi}^2 \cos\theta + mgl\sin\theta - ml^2\ddot{\theta} = 0$$

$$ml^2 \sin^2\theta\ddot{\phi} + 2\dot{\phi}ml^2 \sin\theta\cos\theta = 0$$

20. A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V_2(x)$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this system.

Answer:

I believe there are two errors in the 3rd edition version of this question. Namely, there should be a negative sign infront of $m\dot{x}^2V(x)$ and the $V_2(x)$ should be a $V^2(x)$. Assuming these are all the errors, the solution to this problem goes like this:

$$L = \frac{m^2\dot{x}^4}{12} - m\dot{x}^2 V(x) - V^2(x)$$

Find the equations of motion from Euler-Lagrange formulation.

$$\frac{\partial L}{\partial x} = -m\dot{x}^2 V'(x) - 2V(x)V'(x)$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m^2\dot{x}^2\ddot{x} + 2mV(x)\ddot{x}$$

Thus

$$m\dot{x}^2V' + 2VV' + m^2\dot{x}^2\ddot{x} + 2mV\ddot{x} = 0$$

is our equation of motion. But we want to interpret it. So lets make it look like it has useful terms in it, like kinetic energy and force. This can be done by dividing by 2 and seperating out $\frac{1}{2}mv^2$ and ma's.

$$\frac{m\dot{x}^2}{2}V' + VV' + \frac{m\dot{x}^2}{2}m\ddot{x} + m\ddot{x}V = 0$$

Pull V' terms together and $m\ddot{x}$ terms together:

$$(\frac{m\dot{x}^2}{2} + V)V' + m\ddot{x}(\frac{m\dot{x}^2}{2} + V) = 0$$

Therefore:

$$(\frac{m\dot{x}^2}{2} + V)(m\ddot{x} + V') = 0$$

Now this looks like $E \cdot E' = 0$ because $E = \frac{m\dot{x}^2}{2} + V(x)$. That would mean

$$\frac{d}{dt}E^2 = 2EE' = 0$$

Which allows us to see that E^2 is a constant. If you look at $t = 0$ and the starting energy of the particle, then you will notice that if $E = 0$ at $t = 0$ then $E = 0$ for all other times. If $E \neq 0$ at $t = 0$ then $E \neq 0$ all other times while $m\ddot{x} + V' = 0$.

21. Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until m_1 reaches the hole.)

Answer:

The generalized coordinates for the system are θ, the angle m_1 moves round on the table, and r the length of the string from the hole to m_1. The whole motion of the system can be described by just these coordinates. To write the Lagrangian, we will want the kinetic and potential energies.

$$T = \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -m_2 g(l - r)$$

The kinetic energy is just the addition of both masses, while V is obtained so that $V = -mgl$ when $r = 0$ and so that $V = 0$ when $r = l$.

$$L = T - V = \frac{1}{2}(m_2 + m_1)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 + m_2 g(l - r)$$

To find the Lagrangian equations or equations of motion, solve for each component:

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(m_1 r^2\dot{\theta}) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2\ddot{\theta} + 2m_1 r\dot{r}\dot{\theta} = 0$$

Thus

$$\frac{d}{dt}(m_1 r^2 \dot\theta) = m_1 r(r\ddot\theta + 2\dot\theta\dot r) = 0$$

and

$$\frac{\partial L}{\partial r} = -m_2 g + m_1 r\dot\theta^2$$

$$\frac{\partial L}{\partial \dot r} = (m_2 + m_1)\dot r$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} = (m_2 + m_1)\ddot r$$

Thus

$$m_2 g - m_1 r\dot\theta^2 + (m_2 + m_1)\ddot r = 0$$

Therefore our equations of motion are:

$$\frac{d}{dt}(m_1 r^2 \dot\theta) = m_1 r(r\ddot\theta + 2\dot\theta\dot r) = 0$$

$$m_2 g - m_1 r\dot\theta^2 + (m_2 + m_1)\ddot r = 0$$

See that $m_1 r^2\dot\theta$ is constant. It is angular momentum. Now the Lagrangian can be put in terms of angular momentum. We have $\dot\theta = l/m_1 r^2$.

$$L = \frac{1}{2}(m_1 + m_2)\dot r^2 + \frac{l^2}{2m_1 r^2} - m_2 g r$$

The equation of motion

$$m_2 g - m_1 r\dot\theta^2 + (m_2 + m_1)\ddot r = 0$$

Becomes

$$(m_1 + m_2)\ddot r - \frac{l^2}{m_1 r^3} + m_2 g = 0$$

The problem has been reduced to a single second-order differential equation. The next step is a nice one to notice. If you take the derivative of our new Lagrangian you get our single second-order differential equation of motion.

$$\frac{d}{dt}(\frac{1}{2}(m_1 + m_2)\dot r^2 + \frac{l^2}{2m_1 r^2} - m_2 g r) = (m_1 + m_2)\dot r\ddot r - \frac{l^2}{m_1 r^3}\dot r - m_2 g\dot r = 0$$

$$(m_1 + m_2)\ddot r - \frac{l^2}{m_1 r^3} - m_2 g = 0$$

Thus the first integral of the equation is exactly the Lagrangian. As far as interpreting this, I will venture to say the the Lagrangian is constant, the system is closed, the energy is conversed, the linear and angular momentum are conserved.

22. Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig 1.4, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

Answer:

Add the Lagrangian of the first mass to the Lagrangian of the second mass. For the first mass:

$$T_1 = \frac{1}{2} m l_1^2 \dot{\theta}_1^2$$

$$V_1 = -m_1 g l_1 \cos\theta_1$$

Thus

$$L_1 = T_1 - V_1 = \frac{1}{2} m l_1 \dot{\theta}_1^2 + m g l_1 \cos\theta_1$$

To find the Lagrangian for the second mass, use new coordinates:

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

Then it becomes easier to see the kinetic and potential energies:

$$T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$V_2 = -m_2 g y_2$$

Take derivatives and then plug and chug:

$$T_2 = \frac{1}{2} m_2 (l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 + 2 l_1 l_2 \sin\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \sin^2\theta_2 \dot{\theta}_2^2$$
$$+ l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \cos^2\theta_2 \dot{\theta}_2^2)$$

$$T_2 = \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2)$$

and

$$V_2 = -m g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

Thus

$$L_2 = T_2 - V_2$$
$$= \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2) + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

Add $L_1 + L_2 = L$,

$$L = \frac{1}{2}m_2(l_1^2\dot\theta_1^2+l_2^2\dot\theta_2^2+2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2)+m_2g(l_1\cos\theta_1+l_2\cos\theta_2)+\frac{1}{2}ml_1\dot\theta_1^2+m_1gl_1\cos\theta_1$$

Simplify even though it still is pretty messy:

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2+m_2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2+\frac{1}{2}m_2l_2^2\dot\theta_2^2+(m_1+m_2)gl_1\cos\theta_1+m_2gl_2\cos\theta_2$$

This is the Lagrangian for the double pendulum. To find the equations of motion, apply the usual Euler-Lagrangian equations and turn the crank:

For θ_1:

$$\frac{\partial L}{\partial\theta_1} = -m_2l_1l_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2-(m_1+m_2)gl_1\sin\theta_1$$

$$\frac{\partial L}{\partial\dot\theta_1} = (m_1+m_2)l_2^2\dot\theta_1+m_2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_2$$

$$\frac{d}{dt}[\frac{\partial L}{\partial\dot\theta_1}] = (m_1+m_2)l_1^2\ddot\theta_1+m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_2+\dot\theta_2\frac{d}{dt}[m_2l_1l_2\cos(\theta_1-\theta_2)]$$

Let's solve this annoying derivative term:

$$\dot\theta_2\frac{d}{dt}[m_2l_1l_2\cos(\theta_1-\theta_2)] = \dot\theta_2m_2l_2l_1\frac{d}{dt}[\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2]$$

Using a trig identity,

$$= \dot\theta_2m_2l_2l_1[-\cos\theta_1\sin\theta_2\dot\theta_2-\cos\theta_2\sin\theta_1\dot\theta_1+\sin\theta_1\cos\theta_2\dot\theta_2+\sin\theta_2\cos\theta_1\dot\theta_1]$$

And then more trig identities to put it back together,

$$= \dot\theta_2m_2l_2l_1[\dot\theta_2\sin(\theta_1-\theta_2)-\dot\theta_1\sin(\theta_1-\theta_2)]$$

$$= \dot\theta_2^2m_2l_2l_1\sin(\theta_1-\theta_2)-m_2l_2l_1\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2$$

Plugging this term back into our Euler-Lagrangian formula, the second term of this cancels its positive counterpart:

$$-\frac{\partial L}{\partial\theta}+\frac{d}{dt}[\frac{\partial L}{\partial\dot\theta_1}] = (m_1+m_2)gl_1\sin\theta_1+(m_1+m_2)l_2^2\ddot\theta_1+m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_2+\dot\theta_2^2m_2l_2l_1\sin(\theta_1-\theta_2)$$

Finally, cancel out a l_1 and set to zero for our first equation of motion:

$$(m_1+m_2)g\sin\theta_1+(m_1+m_2)l_1\ddot\theta_1+m_2l_2\cos(\theta_1-\theta_2)\ddot\theta_2+\dot\theta_2^2m_2l_2\sin(\theta_1-\theta_2) = 0$$

Now for θ_2:

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - m_2 g l_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 + m_2 l_2^2 \dot{\theta}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_2 l_1 \cos(\theta_1 - \theta_2) + \dot{\theta}_1[\frac{d}{dt}(m_2 l_2 l_1 \cos(\theta_1 - \theta_2))]$$

Fortunately this is the same derivative term as before, so we can cut to the chase:

$$= m_2 l_2^2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_2 l_1 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1[\dot{\theta}_2 \sin(\theta_1 - \theta_2) - \dot{\theta}_1 \sin(\theta_1 - \theta_2)]$$

Thus

$$-\frac{\partial L}{\partial \theta_2} + \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = +m_2 g l_2 \sin\theta_2 + m_2 l_2^2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_1 l_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 m_2 l_1 l_2 \sin(\theta_1 - \theta_2)$$

Cancel out an l_2 this time, set to zero, and we have our second equation of motion:

$$m_2 g \sin\theta_2 + m_2 l_2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 m_2 l_1 \sin(\theta_1 - \theta_2) = 0$$

Both of the equations of motion together along with the Lagrangian:

$$L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

$$(m_1 + m_2) g \sin\theta_1 + (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \dot{\theta}_2^2 m_2 l_2 \sin(\theta_1 - \theta_2) = 0$$

$$m_2 g \sin\theta_2 + m_2 l_2 \ddot{\theta}_2 + \ddot{\theta}_1 m_2 l_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 m_2 l_1 \sin(\theta_1 - \theta_2) = 0$$

23. Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function $\frac{1}{2}kv^2$ are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for a fall from rest is $v + mg/k$.

Answer:

Work in one dimension, and use the most simple Lagrangian possible:

$$L = \frac{1}{2} m \dot{z}^2 - mgz$$

With dissipation function:

$$F = \frac{1}{2}k\dot{z}^2$$

The Lagrangian formulation is now:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} + \frac{\partial F}{\partial \dot{z}} = 0$$

Plug and chug and get:

$$m\ddot{z} - mg + k\dot{z} = 0$$

Note that at terminal velocity there is no total force acting on you, gravity matches force due to friction, so $m\ddot{z} = 0$:

$$mg = k\dot{z} \quad \rightarrow \quad \dot{z} = \frac{mg}{k}$$

But lets integrate like the problem asks. Let $f = \dot{z} - \frac{mg}{k}$ and substitute into the equation of motion:

$$m\ddot{z} - mg + k\dot{z} = 0$$

$$\frac{m\ddot{z}}{k} - \frac{mg}{k} + \dot{z} = 0$$

$$\frac{m\ddot{z}}{k} + f = 0$$

Note that $f' = \ddot{z}$. Thus

$$\frac{mf'}{k} + f = 0$$

$$\frac{f'}{f} = -\frac{k}{m}$$

$$\ln f = -\frac{k}{m}t + C$$

$$f = Ce^{-\frac{k}{m}t}$$

Therefore

$$\dot{z} - \frac{mg}{k} = Ce^{-\frac{k}{m}t}$$

Plugging in the boundary conditions, that at $t = 0$, $\dot{z} = 0$, we solve for C

$$-\frac{mg}{k} = C$$

Thus

$$\dot{z} - \frac{mg}{k} = -\frac{mg}{k}e^{-\frac{k}{m}t}$$

and with $t \to \infty$ we have finally

$$\dot{z} = \frac{mg}{k}$$

24. A spring of rest length L_a (no tension) is connected to a support at one end and has a mass M attached at the other. Neglect the mass of the spring, the dimension of the mass M, and assume that the motion is confined to a vertical plane. Also, assume that the spring only stretches without bending but it can swing in the plane.

1. Using the angular displacement of the mass from the vertical and the length that the string has stretched from its rest length (hanging with the mass m), find Lagrange's equations.
2. Solve these equations fro small stretching and angular displacements.
3. Solve the equations in part (1) to the next order in both stretching and angular displacement. This part is amenable to hand calculations. Using some reasonable assumptions about the spring constant, the mass, and the rest length, discuss the motion. Is a resonance likely under the assumptions stated in the problem?
4. (For analytic computer programs.) Consider the spring to have a total mass $m << M$. Neglecting the bending of the spring, set up Lagrange's equations correctly to first order in m and the angular and linear displacements.
5. (For numerical computer analysis.) Make sets of reasonable assumptions of the constants in part (1) and make a single plot of the two coordinates as functions of time.

Answer:

This is a spring-pendulum. It's kinetic energy is due to translation only.

$$T = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2)$$

The more general form of v is derived in problem 15 if this step was not clear. Just disregard ϕ direction. Here r signifies the total length of the spring, from support to mass at any time.

As in problem 22, the potential has a term dependent on gravity, but it also has the potential of your normal spring.

$$V = -mgr\cos\theta + \frac{1}{2}k(r - L_a)^2$$

Note that the potential due to gravity depends on the total length of the spring, while the potential due to the spring is only dependent on the stretching from its natural length. Solving for the Lagrangian:

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + mgr\cos\theta - \frac{1}{2}k(r - L_a)^2$$

Lets solve for Lagrange's equations now.

For r:

$$\frac{\partial L}{\partial r} = mg\cos\theta + mr\dot{\theta}^2 - k(r - L_a)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

For θ:

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

Bring all the pieces together to form the equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + k(r - L_a) - mg\cos\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$$

For part b, we are to solve these equations for small stretching and angular displacements. Simplify the equations above by canceling out m's, r's and substituting θ for $\sin\theta$, and 1 for $\cos\theta$.

$$\ddot{r} - r\dot{\theta}^2 + \frac{k}{m}(r - L_a) - g = 0$$

$$\ddot{\theta} + \frac{2\dot{r}}{r}\dot{\theta} + \frac{g}{r}\theta = 0$$

Solve the first equation, for r, with the initial condition that $\theta_0 = 0$, $\dot{\theta}_0 = 0$, $r_0 = 0$ and $\dot{r}_0 = 0$:

$$r = L_a + \frac{mg}{k}$$

Solve the second equation, for θ, with the same initial conditions:

$$\theta = 0$$

This is the solution of the Lagrangian equations that make the generalized force identically zero. To solve the next order, change variables to measure deviation from equilibrium.

$$x = r - (L_a + \frac{mg}{k}), \quad \theta$$

Substitute the variables, keep only terms to 1st order in x and θ and the solution is:

$$\ddot{x} = -\frac{k}{m}x \quad \ddot{\theta} = -\frac{g}{L_a + \frac{m}{k}g}\theta$$

In terms of the original coordinates r and θ, the solutions to these are:

$$r = L_a + \frac{mg}{k} + A\cos(\sqrt{\frac{k}{m}}t + \phi)$$

$$\theta = B\cos(\sqrt{\frac{kg}{kL_a + mg}}t + \phi')$$

The phase angles, ϕ and ϕ', and amplitudes A and B are constants of integration and fixed by the initial conditions. Resonance is very unlikely with this system. The spring pendulum is known for its nonlinearity and studies in chaos theory.