General Relativity Exam

Michael Good

April 25, 2006

Problem 3

Taking Robertson-Walker metric

$$ds^2 = -dt^2 + R(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

with $k = +1, 0, -1$ and $R(t)$ the cosmological scale. Use $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and $T^{\mu\nu}{}_{;\nu} = 0$ to obtain the Friedmann equation

$$\dot{R}^2 + k = \frac{8\pi}{3}\rho R^2$$

and the equation of energy conservation

$$\frac{d}{dR}(\rho R^3) = -3pR^2$$

assuming a perfect fluid stress enrgy tensor where $\rho(t)$ is the matter total energy density and p is the isotropic pressure.

Specialize to the case of a flat universe $k = 0$ and compute the resulting time dependence of the scale factor when the universe is filled with dust $p = 0$ and in the case where the universe is filled with radiation $p = \rho/3$.

Solution:

Writing Einstein's equation in the form

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

We see that we need the Ricci tensor and the stress-energy tensor for a perfect fluid. The Ricci tensor involves calculating the Christoffel symbols from the metric which is a tedious but easy calculation. The stress-energy tensor can be found more quickly. As our fluid is perfect, the fluid may be at rest in comoving coordinates with a 4-velocity of $U^\mu = (1, 0, 0, 0)$ and the stress-energy tensor $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$ is, with an index raised:

$$T^{\mu}_{\nu} = diag(-\rho, p, p, p)$$

with a trace of

$$T = -\rho + 3p$$

We will remember this for later. For now, lets focus on the Ricci tensor. To find the Christoffel symbols, simply use:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu})$$

Lets explicitly calculate a few. Try Γ^3_{03}.

$$\Gamma^3_{03} = \frac{1}{2} g^{3\rho}(\partial_0 g_{3\rho} + \partial_3 g_{\rho 0} - \partial_{\rho} g_{03})$$

This is just the first term and only with $\rho = 3$, as the others vanish

$$\Gamma^3_{03} = \frac{1}{2} g^{33} \partial_0 g_{33} = \frac{1}{2R^2}(\partial_t R^2) = \frac{\dot{R}}{R}$$

Lets try another:

$$\Gamma^0_{22} = \frac{1}{2} g^{00}(-\partial_0 g_{22}) = \frac{1}{2} \partial_t (R^2 r^2) = \frac{r^2}{2}(\partial_t R^2) = r^2 R\dot{R}$$

The rest follow in exactly the same way. Here are all of them:

$\Gamma^0_{11} = R\dot{R}/(1 - kr^2)$	$\Gamma^0_{22} = R\dot{R}r^2$	$\Gamma^0_{33} = R\dot{R}r^2 \sin^2\theta$
$\Gamma^1_{01} = \dot{R}/R$	$\Gamma^1_{11} = kr/(1 - kr^2)$	$\Gamma^1_{22} = -r(1 - kr^2)$
$\Gamma^1_{33} = -r(1 - kr^2)\sin^2\theta$	$\Gamma^2_{12}, \Gamma^3_{13} = 1/r$	$\Gamma^3_{23} = \cot\theta$
$\Gamma^2_{02} = \dot{R}/R$	$\Gamma^3_{03} = \dot{R}/R$	$\Gamma^2_{33} = -\sin\theta\cos\theta$

With our Christoffel symbols in hand, we are in a position to calculate the Ricci tensor from:

$$R_{\sigma\nu} = \partial_{\lambda}\Gamma^{\lambda}_{\nu\sigma} - \partial_{\nu}\Gamma^{\lambda}_{\lambda\sigma} + \Gamma^{\lambda}_{\lambda\mu}\Gamma^{\mu}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\mu}_{\lambda\sigma}$$

An important shortcut is that because our space is isotropic, we only care about two equations from the Ricci tensor, that is the tt and one of the space-space ones, I'll go with just 22. I'll explicitly go through the tt one.

$$R_{00} = -\Gamma^{\sigma}_{0\sigma,0} - \Gamma^{\rho}_{0\sigma}\Gamma^{\sigma}_{\rho 0}$$

Plugging in the appropriate Γ's we get:

$$R_{00} = -3\partial_t(\frac{\dot{R}}{R}) - 3(\frac{\dot{R}}{R})^2 = -3\frac{\ddot{R}}{R} + 3\frac{\dot{R}}{R^2}\dot{R} - 3(\frac{\dot{R}}{R^2})^2 = -3\frac{\ddot{R}}{R}$$

The R_{22} works exactly the same way and is:

$$R_{22} = r^2(R\ddot{R} + 2\dot{R}^2 + 2k)$$

Remember the include all the contraction terms, as it can be easy to miss some. Now, back to our Einstein field equations. We still have

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

and lets see what they give us for our tt term.

$$R_{00} = 8\pi(T_{00} - \frac{1}{2}g_{00}T)$$

Since $T_{00} = \rho$ and $T = -\rho + 3p$ and bringing R_{00} down from above, we have:

$$-3\frac{\ddot{R}}{R} = 8\pi(\rho + \frac{1}{2}(-\rho + 3p)) = 4\pi(\rho + 3p) \tag{1}$$

So much for that. How about our 22 term? Bringing R_{22} down from above and using $g_22 = R^2r^2$, $T_{22} = g_22p = R^2r^2p$, we get

$$r^2(R\ddot{R} + 2\dot{R}^2 + 2k) = 8\pi(R^2r^2p - \frac{1}{2}R^2r^2(3p - \rho))$$

Take out the silly r^2, divide by R and collect the ρ and p's.

$$\frac{\ddot{R}}{R} + 2(\frac{\dot{R}}{R})^2 + 2\frac{k}{R^2} = 4\pi(\rho - p) \tag{2}$$

Combine equations (1) and (2) to get rid of the second derivative.

$$-\frac{4\pi}{3}(\rho + 3p) + 2(\frac{\dot{R}}{R})^2 + 2\frac{k}{R^2} = 4\pi(\rho - p)$$

Simplified, the p's cancel:

$$2(\frac{\dot{R}}{R})^2 + 2\frac{k}{R^2} = \frac{16\pi}{3}\rho$$

$$\boxed{\dot{R}^2 + k = \tfrac{8\pi}{3}\rho R^2} \tag{3}$$

This is the Friedmann equation. As for the equation of energy conservation, use this trick:

$$\frac{1}{2}\frac{d}{dR}\dot{R}^2 = \ddot{R} \tag{4}$$

You can see that this checks because

$$\dot{R}\frac{d\dot{R}}{dR} = \frac{dR}{dt}\frac{d\dot{R}}{dR} = \ddot{R}$$

With this, we rearrange equation (1):

$$\ddot{R} = -\frac{4}{3}\pi(\rho + 3p)R$$

and use our trick, equation (4).

$$\frac{1}{2}\frac{d}{dR}\dot{R}^2 = -\frac{4}{3}\pi(\rho + 3p)R$$

and throw in $\dot{R}$ from equation (2):

$$\frac{1}{2}\frac{d}{dR}(\frac{8\pi}{3}\rho R^2 - k) = -\frac{4}{3}\pi(\rho + 3p)R$$

This is

$$\frac{d}{dR}(\rho R^2) = -(\rho + 3p)R$$

We can move the first term on the right side over the left hand side and multiply the whole equation by R:

$$R\frac{d}{dR}(\rho R^2) + \rho R^2 = -3pR^2$$

Note that this is just the product rule, so bring it together and we have:

$$\boxed{\tfrac{d}{dR}(\rho R^3) = -3pR^2} \tag{5}$$

This is the energy conservation equation.

Specializing to the case of a flat universe, I am now interested in the evolution of the scale factor. We numerically integrate the Friedmann equation to find the evolution of the scale factor. The types of solutions for the different parameters may be examined in this way. If we imagine that all the different components of the energy density ρ_i, evolve as power laws and we have $\rho_i = \rho_{i0}R^{-n_i}$. The behavior of sources are: matter ($n_i = 3$), radiation ($n_i = 4$), curvature ($n_i = 2$) and vacuum ($n_i = 0$). So we have

$$\rho \propto R^{-n}$$

In flat space, $k = 0$ and the Friedmann equation gives

$$\dot{R}^2 \propto \rho R^2$$

$$\dot{R} \propto R^{-n}R^2 = R^{1-n/2}$$

$$\frac{dR}{R^{1-n/2}} \propto dt$$

The dependence on time, for when the universe is filled with dust ($p = 0$), ($n = 3$), can be found by

$$R^{1/2} dR \propto dt$$

which, integrated is:

$$\boxed{R \propto t^{2/3} \text{ for } p = 0} \tag{6}$$

For the case of a universe filled with radiation ($p = \frac{\rho}{3}$), ($n = 4$), we can find time dependence by:

$$R dR \propto dt$$

$$\boxed{R \propto t^{1/2} \text{ for } p = \frac{\rho}{3}} \tag{7}$$

We would only consider flat universes dominated by a single source, or completely empty universes with spatial curvature because we are including spatial curvature as an effective energy source. Generally the relationship goes like:

$$R \propto t^{2/n} \tag{8}$$

for ($\rho \propto a^{-n}$).

Problem 4

Work out the equations of stellar equilbrium for relativistic spherically symmetric star. Take the metric to have "standard" form in spherical symmetry.

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2$$

Assume the star is described by a perfect fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$$

with ρ the total energy density, p the pressure, and U^μ the four velocity, which satsfies $g_{\mu\nu}U^\mu U^\nu = -1$. In this coordinate system the fluid in the star will be at rest.

Use Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and the equations of motion of the matter, hydrostatic equilibrium, $T^{\mu\nu}{}_{;\nu} = 0$ to find the differential equations for the metric and the pressure gradient. Show

$$A(r) = \frac{1}{1 - \frac{2M(r)}{r}}$$

with

$$M(r) = 4\pi \int_0^r dr' r'^2 \rho(r')$$

and

$$\frac{1}{B}\frac{dB}{dr} = -\frac{2}{\rho + p}\frac{dp}{dr}$$

and the Oppenheimer-Volkoff equation

$$\frac{dp}{dr} = -\frac{(\rho + p)(M(r) + 4\pi r^3 p)}{r^2 - 2M(r)r}$$

Solution:

Armed with Einstein's field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

we see we must now solve for the Ricci tensor, the Ricci scalar, and the stress-energy tensor. As before, the stress energy tensor describes a perfect fluid only this time we have a different and more general form for the metric. The stress energy tensor assumes the form:

$$T_{\mu\nu} = \begin{pmatrix} B\rho & & & \\ & Ap & & \\ & & r^2 p & \\ & & & r^2 \sin^2\theta \; p \end{pmatrix}$$

We are good to go on the right hand side of Einstein's equations, now let's look at the left hand side. From the metric you've got to determine the Christoffel symbols. Here is Γ^t_{tr}:

$$\Gamma^t_{tr} = \frac{1}{2} g^{t\rho}(\partial_t g_{r\rho} + \partial_r g_{\rho t} - \partial_\rho g_{tr})$$

Only the first term survives

$$\Gamma^t_{tr} = \frac{1}{2} g^{tt} \partial_r g_{tt} = \frac{1}{2B} \partial_r B$$

Calculating all the Christoffel symbols as in Problem 3, I obtain:

$$\begin{array}{lll} \Gamma^t_{tr} = \frac{1}{2}\partial_r lnB & \Gamma^r_{tt} = \frac{B}{2A}\partial_r lnB & \Gamma^r_{rr} = \frac{1}{2}\partial_r lnA \\ \Gamma^\theta_{r\theta} = \frac{1}{r} & \Gamma^r_{\theta\theta} = \frac{-r}{A} & \Gamma^\phi_{r\phi} = \frac{1}{r} \\ \Gamma^r_{\phi\phi} = \frac{-r}{A}\sin^2\theta & \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^\phi_{\theta\phi} = \cot\theta \end{array}$$

From here I solve for the Ricci tensor using:

$$R_{\sigma\nu} = \partial_\lambda \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\lambda\sigma} + \Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{\nu\sigma} - \Gamma^\lambda_{\nu\mu}\Gamma^\mu_{\lambda\sigma}$$

Allow me to solve for the G_{tt} component of Einstein's equations. My Ricci tensor is:

$$R_{tt} = \frac{B}{A}[\frac{1}{2}\partial_r^2 lnB + (\frac{1}{2}\partial_r lnB)^2 - \frac{1}{2}\partial_r lnB \frac{1}{2}\partial_r lnA + \frac{1}{r}\partial_r lnB]$$

My Ricci scalar is:

$$R = -\frac{2}{A}[\frac{1}{2}\partial_r^2 lnB + (\frac{1}{2}\partial_r lnB)^2 - \frac{1}{2}\partial_r lnB \frac{1}{2}\partial_r lnA + \frac{1}{r}(\partial_r lnB - \partial_r lnA) + \frac{1}{r^2}(1-A)]$$

Solving for $-\frac{1}{2}Rg_{tt}$ gives me my G_{tt}

$$G_{tt} = R_{tt} - \frac{1}{2}Rg_{tt} = \frac{B}{A}[\frac{1}{r}\partial_r lnA - \frac{1}{r^2} - \frac{A}{r^2}]$$

Setting this equal to $8\pi T_{tt}$ with $T_{tt} = B\rho$, yields a final equation for the tt component:

$$\frac{1}{Ar^2}(r\partial_r lnA - 1 + A) = 8\pi\rho \tag{9}$$

This is handy as it has only A and ρ in it. At this point I will replace A by a new function $M(r)$ which I will just denote as $M(r) = m$:

$$m = \frac{1}{2}(r - \frac{r}{A})$$

This is equivalent to

$$\boxed{A = [1 - \tfrac{2m}{r}]^{-1}} \tag{10}$$

Does this check? Plugging my new function into my tt equation yields:

$$\frac{1}{r^2}[1 - \frac{2m}{r}][-r\partial_r(ln(1 - \frac{2m}{r}) - 1 + (1 - \frac{2m}{r})^{-1}] = 8\pi\rho$$

$$\frac{1}{r^2}[1 - \frac{2m}{r}][-r(1 - \frac{2m}{r})^{-1}\partial_r(-\frac{2m}{r}) - 1 + (1 - \frac{2m}{r})^{-1}] = 8\pi\rho$$

$$-\frac{1}{r}\partial_r(-\frac{2m}{r}) - \frac{1}{r^2}(1 - \frac{2m}{r}) + \frac{1}{r^2} = 8\pi\rho$$

$$-\frac{1}{r}[2mr^{-2} - r^{-1}2\frac{dm}{dr}] + \frac{2m}{r^3} = 8\pi\rho$$

$$-\frac{2m}{r^3} + \frac{2}{r^2}\frac{dm}{dr} + \frac{2m}{r^3} = 8\pi\rho$$

This is

$$\frac{dm}{dr} = 4\pi\rho r^2$$

Integrate and

$$\boxed{m(r) = 4\pi\int_0^r \rho(r')r'^2 dr'} \tag{11}$$

it checks.

My R_{rr} piece is

$$R_{rr} = -\frac{1}{2}\partial_r^2 lnB - (\frac{1}{2}\partial_r lnB)^2 + \frac{1}{2}\partial_r lnB\frac{1}{2}\partial_r lnA + \frac{1}{r}\partial_r lnA$$

Using the same Ricci scalar above, I solve for

$$G_{rr} = R_{rr} - \frac{1}{2}g_{rr}R$$

which gives me

$$G_{rr} = \frac{1}{r^2}(r\partial_r lnB + 1 - A)$$

Setting this equal to $8\pi T_{rr} = 8\pi\rho A$ leaves us with another grand equation:

$$\frac{1}{r^2}(r\partial_r lnB + 1 - A) = 8\pi\rho A$$

$$\frac{1}{B}\frac{dB}{dr} = 8\pi\rho r A - \frac{1}{r} + \frac{A}{r} \tag{12}$$

which is the rr component. Plugging in our A we get

$$\frac{1}{B}\frac{dB}{dr} = \frac{8\pi\rho r^2}{r-2m} - \frac{1}{r} + \frac{1}{r-2m}$$

Bring together like terms, and multiply the second term on the right hand side by $1 = A/A$

$$\frac{1}{B}\frac{dB}{dr} = \frac{8\pi\rho r^3 + 2m}{r^2 - 2mr} \tag{13}$$

Save this for latter, as we'll need this modified rr term to get the Oppenheimer-Volkoff equation. At this point I want to use the equations of motion of the matter to solve for $\frac{1}{B}\frac{dB}{dr}$. Evaluating $T^{\mu\nu}{}_{;\nu} = 0$ is most easily done by remembering Euler's equation,

$$(\rho + p)U_{\mu;\nu}U^{\nu} = -p_{,\mu} - p_{,\nu}U^{\nu}U_{\mu}$$

The only part that contributes is $\nu = r$ so we have

$$-\frac{dp}{dr} = (\rho + p)U_{r;\nu}U^{\nu} + 0$$

$$-\frac{dp}{dr} = -(\rho + p)\Gamma^{\lambda}_{r\nu}U_{\lambda}U^{\nu} = -(\rho + p)\Gamma^{t}_{rt}U_tU^t = (\rho + p)\frac{1}{2}\partial_r lnB$$

So we've got

$$-\frac{dp}{dr} = \frac{(\rho + p)}{2}\frac{1}{B}\frac{dB}{dr}$$

This is

$$\boxed{\frac{1}{B}\frac{dB}{dr} = -\frac{2}{\rho + p}\frac{dp}{dr}} \tag{14}$$

Take equation (13) and (14) and combine:

$$-\frac{2}{\rho + p}\frac{dp}{dr} = \frac{8\pi\rho r^3 + 2m}{r^2 - 2mr}$$

Isolate the pressure gradient, noting that $m = M(r)$ still.

$$\boxed{\frac{dp}{dr} = -\frac{(\rho+p)(4\pi r^3 p + m)}{r^2 - 2mr}} \tag{15}$$

This is the Oppenheimer- Volkoff equation, or just simply, the equation of hydrostatic equilibrium.