Homework 3: # 2.13, 2.14

Michael Good

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2.13 A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of the Lagrange's undetermined multipliers and Lagrange's equations. Find the height at which the particle falls off.

Answer:

The Lagrangian is

$$L = T - V \qquad \Rightarrow \qquad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

Where r is the distance the particle is away from the center of the hoop. The particle will eventually fall off but while its on the hoop, r will equal the radius of the hoop, a. This will be the constraint on the particle. Here when $\theta = 0$, (at the top of the hoop) potential energy is mgr, and when $\theta = 90^o$ (at half of the hoop) potential energy is zero. Using Lagrange's equations with undetermined multipliers,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda \frac{\partial f_k}{\partial q_j} = 0$$

with our equation of constraint, $f = r = a$ as long as the particle is touching the hoop. Solving for the motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta$$

$$\lambda\frac{\partial f_r}{\partial r} = \lambda * 1$$

thus

$$-m\ddot{r} + mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

solving for the other equation of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = mr^2\ddot\theta + 2mr\dot r\dot\theta$$
$$\frac{\partial L}{\partial \theta} = mgr\sin\theta$$
$$\lambda\frac{\partial f_\theta}{\partial \theta} = \lambda * 0$$

thus

$$-mr^2\ddot\theta - 2mr\dot r\dot\theta + mgr\sin\theta = 0$$

The equations of motion together are:

$$-m\ddot r + mr\dot\theta^2 - mg\cos\theta + \lambda = 0$$

$$-mr^2\ddot\theta - 2mr\dot r\dot\theta + mgr\sin\theta = 0$$

To find the height at which the particle drops off, λ can be found in terms of θ. The force of constraint is λ and $\lambda = 0$ when the particle is no longer under the influence of the force of the hoop. So finding λ in terms of θ and setting λ to zero will give us the magic angle that the particle falls off. With the angle we can find the height above the ground or above the center of the hoop that the particle stops maintaining contact with the hoop.

With the constraint, the equations of motion become,

$$ma\dot\theta^2 - mg\cos\theta + \lambda = 0$$
$$-ma^2\dot\theta + mga\sin\theta = 0$$

Solving for θ, the m's cancel and 1 a cancels, we are left with

$$\frac{g}{a}\sin\theta = \ddot\theta$$

solving this and noting that

$$\dot\theta d\dot\theta = \ddot\theta d\theta$$

by the 'conservation of dots' law Engel has mentioned :), or by

$$\ddot\theta = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot\theta}{dt} = \frac{d\dot\theta}{d\theta}\frac{d\theta}{dt} = \dot\theta\frac{d\dot\theta}{d\theta}$$
$$\int \frac{g}{a}\sin\theta d\theta = \int \dot\theta d\dot\theta$$

$$-\frac{g}{a}\cos\theta = \frac{\dot\theta^2}{2} + constant$$

The constant is easily found because at the top of the hoop, $\theta = 0$ and $\dot\theta = 0$ at $t = 0$ so,

$$-\frac{2g}{a}\cos\theta + \frac{2g}{a} = \dot{\theta}^2$$

Plug this into our first equation of motion to get an equation dependent only on θ and λ

$$ma[-\frac{2g}{a}\cos\theta + \frac{2g}{a}] - mg\cos\theta = -\lambda$$

$$-3mg\cos\theta + 2mg = -\lambda$$

Setting $\lambda = 0$ because this is at the point where the particle feels no force from the hoop, and θ_0 equals

$$\theta_0 = \cos^{-1}(\frac{2}{3}) = 48.2^o$$

And if our origin is at the center of the hope, then the height that it stops touching the hoop is just $R\cos\theta_0$ or

$$h = R\cos(\cos^{-1}\frac{2}{3}) = \frac{2}{3}R$$

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If we say the hoop is a fully circular and somehow fixed with the origin at the bottom of the hoop, then we have just moved down by R and the new height is

$$H = R + \frac{2}{3}R = \frac{5}{3}R$$

2.14 A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in figure. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder.

Answer:

Two equations of constraint:

$$\rho = r + R \qquad r(\phi - \theta) = R\theta$$

My generalized coordinates are ρ, θ, and ϕ. The first equation comes from the fact that as long as the hoop is touching the cylinder the center of mass of the hoop is exactly $r + R$ away from the center of the cylinder. I'm calling it f_1. The second one comes from no slipping:

$$r\phi = s \quad \rightarrow \quad s = (R + r)\theta$$

$$r\phi - r\theta = R\theta$$

$$r(\phi - \theta) = R\theta$$

Where θ is the angle ρ makes with the vertical and ϕ is the angle r makes with the vertical. I'm calling this equation f_2.

$$f_1 = \rho - r - R = 0 \qquad f_2 = R\theta - r\phi + r\theta = 0$$

The Lagrangian is $T - V$ where T is the kinetic energy of the hoop about the cylinder and the kinetic energy of the hoop about its center of mass. The potential energy is the height above the center of the cylinder. Therefore

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + r^2\dot{\phi}^2) - mg\rho\cos\theta$$

Solving for the equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = \sum_k \lambda_k \frac{\partial f_k}{\partial \rho}$$

$$m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\cos\theta = \lambda_1\frac{\partial f_1}{\partial \rho} + \lambda_2\frac{\partial f_2}{\partial \rho}$$

$$m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\cos\theta = \lambda_1 \tag{1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \sum_k \lambda_k \frac{\partial f_k}{\partial \theta}$$

$$\frac{d}{dt}(m\rho^2\dot{\theta}) - (-mg\rho\sin\theta) = \lambda_1\frac{\partial f_1}{\partial \theta} + \lambda_2\frac{\partial f_2}{\partial \theta}$$

$$m\rho^2\ddot{\theta} + \dot{\theta}2m\rho\dot{\rho} + mg\rho\sin\theta = \lambda_1(0) + \lambda_2(R+r)$$

$$m\rho^2\ddot{\theta} + 2m\rho\dot{\rho}\dot{\theta} + mg\rho\sin\theta = \lambda_2(R+r) \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \sum_k \lambda_k \frac{\partial f_k}{\partial \phi}$$

$$\frac{d}{dt}(mr^2\dot{\phi}) - 0 = \lambda_1\frac{\partial f_1}{\partial \phi} + \lambda_2\frac{\partial f_2}{\partial \phi}$$

$$mr^2\ddot{\phi} = -\lambda_2 r \tag{3}$$

I want the angle θ. This will tell me the point that the hoop drops off the cylinder. So I'm going to apply the constraints to my equations of motion, attempt to get an equation for θ, and then set λ_1 equal to zero because that will be when the force of the cylinder on the hoop is zero. This will tell me the value of θ. Looking for an equation in terms of only θ and λ_1 will put me in the right position.
The constraints tell me:

$$\rho = r + R \quad \rightarrow \quad \dot{\rho} = \ddot{\rho} = 0$$

$$\phi = \frac{R+r}{r}\theta \quad \rightarrow \quad \dot{\phi} = \frac{R+r}{r}\dot{\theta} \quad \rightarrow \quad \ddot{\phi} = \frac{R+r}{r}\ddot{\theta}$$

Solving (3) using the constraints,

$$mr^2\ddot{\phi} = -\lambda_2 r$$

$$\ddot{\theta} = -\frac{\lambda_2}{m(R+r)} \tag{4}$$

Solving (2) using the constraints,

$$m(R+r)\ddot{\theta} + mg\sin\theta = \lambda_2$$

$$\ddot{\theta} = \frac{\lambda_2 - mg\sin\theta}{m(R+r)} \tag{5}$$

Setting (4) = (5)

$$-\lambda_2 = \lambda_2 - mg\sin\theta$$

$$\lambda_2 = \frac{mg}{2}\sin\theta \tag{6}$$

Plugging (6) into (4) yields a differential equation for θ

$$\ddot{\theta} = \frac{-g}{2(R+r)}\sin\theta$$

If I solve this for $\dot{\theta}^2$ I can place it in equation of motion (1) and have an expression in terms of θ and λ_1. This differential equation can be solved by trying this:

$$\dot{\theta}^2 = A + B\cos\theta$$

Taking the derivative,

$$2\dot{\theta}\ddot{\theta} = -B\sin\theta\dot{\theta}$$

$$\ddot{\theta} = -\frac{B}{2}\sin\theta$$

Thus

$$B = -\frac{q}{R+r}$$

From initial conditions, $\theta = 0$, $\dot{\theta} = 0$ at $t = 0$ we have A:

$$A = -B \quad \rightarrow \quad A = \frac{q}{R+r}$$

Therefore

$$\dot{\theta}^2 = \frac{q}{R+r} - \frac{q}{R+r}\cos\theta$$

Now we are in a position to plug this into equation of motion (1) and have the equation in terms of θ and λ_1

$$-m(R+r)(\frac{q}{R+r} - \frac{q}{R+r}\cos\theta) + mg\cos\theta = \lambda_1$$

$$-mg + 2mg\cos\theta = \lambda_1$$

$$mg(2\cos\theta - 1) = \lambda_1$$

Setting the force of constraint equal to zero will give us the angle that the hoop no longer feels a force from the cylinder:

$$2\cos\theta_0 - 1 = 0$$

$$\cos\theta_0 = \frac{1}{2} \quad \rightarrow \quad \theta_0 = 60^o$$

With our origin at the center of the cylinder, the height that the center of mass of the hoop falls off is

$$h_{cm} = \rho\cos(60^o) = \frac{1}{2}\rho$$

Or if you prefer the height that the hoop's surface stops contact with cylinder:

$$h = \frac{1}{2}R$$

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