Homework 12: # 10.13, 10.27, Cylinder

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10.13
A particle moves in periodic motion in one dimension under the influence of a potential $V(x) = F|x|$, where F is a constant. Using action-angle variables, find the period of the motion as a function of the particle's energy.

Solution:

Define the Hamiltonian of the particle

$$H \equiv E = \frac{p^2}{2m} + F|q|$$

Using the action variable definition, which is Goldstein's (10.82):

$$J = \oint p\, dq$$

we have

$$J = \oint \sqrt{2m(E - Fq)}\, dq$$

For F is greater than zero, we have only the first quadrant, integrated from $q = 0$ to $q = E/F$ (where $p = 0$). Multiply this by 4 for all of phase space and our action variable J becomes

$$J = 4\int_0^{E/F} \sqrt{2m}\sqrt{E - Fq}\, dq$$

A lovely u-substitution helps out nicely here.

$$u = E - Fq \qquad \rightarrow \qquad du = -F\, dq$$

$$J = 4\int_E^0 \sqrt{2m} u^{1/2} \frac{1}{-F}\, du$$

$$J = \frac{4\sqrt{2m}}{F}\int_0^E u^{1/2}\, du = \frac{8\sqrt{2m}}{3F}E^{3/2}$$

Goldstein's (10.95) may help us remember that

$$\frac{\partial H}{\partial J} = \nu$$

and because $E = H$ and $\tau = 1/\nu$,

$$\tau = \frac{\partial J}{\partial E}$$

This is

$$\tau = \frac{\partial}{\partial E}[\frac{8\sqrt{2m}}{3F}E^{3/2}]$$

And our period is

$$\tau = \frac{4\sqrt{2mE}}{F}$$

10.27 Describe the phenomenon of small radial oscillations about steady circular motion in a central force potential as a one-dimensional problem in the action-angle formalism. With a suitable Taylor series expansion of the potential, find the period of the small oscillations. Express the motion in terms of J and its conjugate angle variable.

Solution:

As a reminder, Taylor series go like

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f^{''}(a) + ...$$

Lets expand around some r_0 for our potential,

$$U(r) = U(r_0) + (r-r_0)U'(r_0) + \frac{1}{2}(r-r_0)^2 U^{''}(r_0) + ...$$

Using the form of the Hamiltonian, involving two degrees of freedom, in polar coordinates,(eq'n 10.65) we have

$$H = \frac{1}{2m}(p_r^2 + \frac{l^2}{r^2}) + V(r)$$

Defining a new equivalent potential, $U(r)$ the Hamiltonian becomes

$$H = \frac{1}{2m}p_r^2 + U(r)$$

The r_0 from above will be some minimum of $U(r)$,

$$U'(r_0) = -\frac{l^2}{mr_0^3} + V'(r_0) = 0$$

The second derivative is the only contribution

$$U'' = \frac{3l^2}{mr_0^4} + V''(r_0) = k$$

where $k > 0$ because we are at a minimum that is concave up. If there is a small oscillation about circular motion we may let

$$r = r_0 + \lambda$$

where λ will be very small compared to r_0. Thus our Hamiltonian becomes

$$H = \frac{1}{2m}p_r^2 + U(r_0 + \lambda)$$

This is

$$H = \frac{1}{2m}p_r^2 + U(r_0) + \frac{1}{2}(r - r_0)^2 U''(r_0)$$

$$H = \frac{1}{2m}p_r^2 + U(r_0) + \frac{1}{2}\lambda^2 U''(r_0) = E$$

If we use the small energy ϵ defined as

$$\epsilon = E - U(r_0)$$

We see

$$\epsilon = \frac{1}{2m}p_r^2 + \frac{1}{2}\lambda^2 k$$

This energy is the effect on the frequency, so following section 10.6

$$\epsilon = \frac{J\omega}{2\pi}$$

We have for the action variable

$$J = 2\pi\epsilon\sqrt{\frac{m}{k}}$$

and a period

$$\tau = \frac{\partial J}{\partial \epsilon} = 2\pi\sqrt{\frac{m}{k}}$$

with motion given by

$$r = r_0 + \sqrt{\frac{J}{\pi m\omega}}\sin 2\pi\omega$$

$$p_r = \sqrt{\frac{mJ\omega}{\pi}}\cos 2\pi\omega$$

A particle is constrained to the edge of a cylinder. It is released and bounces around the perimeter. Find the two frequencies of its motion using the action angle variable formulation.

Solution:

Trivially, we know the frequency around the cylinder to be its angular speed divided by 2π because it goes 2π radians in one revolution.

$$\nu_\theta = \frac{\dot{\theta}}{2\pi}$$

And also simply, we may find the frequency of its up and down bouncing through Newtonian's equation of motion.

$$h = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}}$$

Multiply this by 2 because the period will be measured from a point on the bottom of the cylinder to when it next hits the bottom of the cylinder again. The time it takes to fall is the same time it takes to bounce up, by symmetry.

$$T = 2\sqrt{\frac{2h}{g}} \quad \rightarrow \quad \nu_z = \frac{1}{2}\sqrt{\frac{g}{2h}}$$

To derive these frequencies via the action-angle formulation we first start by writing down the Hamiltonian for the particle.

$$H \equiv E = mgz + \frac{p_z^2}{2m} + \frac{p_\theta^2}{2mR^2}$$

Noting that p_θ is constant because there is no external forces on the system, and because θ does not appear in the Hamiltonian, therefore it is cyclic and its conjugate momentum is constant.

$$p_\theta = m\dot{\theta}R^2$$

we may write

$$J_\theta = 2\pi p_\theta$$

based on Goldstein's (10.101), and his very fine explanation. Breaking the energy into two parts, one for θ and one for z, we may express the E_θ part as a function of J_θ.

$$E_\theta = \frac{p_\theta^2}{2mR^2} = \frac{J_\theta^2}{4\pi^2 2mR^2}$$

The frequency is

$$\nu_\theta = \frac{\partial E_\theta}{\partial J_\theta} = \frac{J_\theta}{4\pi^2 m R^2}$$

$$\nu_\theta = \frac{J_\theta}{4\pi^2 m R^2} = \frac{2\pi p_\theta}{4\pi^2 m R^2} = \frac{p_\theta}{2\pi m R^2} = \frac{m\dot{\theta} R^2}{2\pi m R^2}$$

Thus we have

$$\nu_\theta = \frac{\dot{\theta}}{2\pi}$$

The second part is a bit more involved algebraically. Expressing the energy for z:

$$E_z = mgz + \frac{p_z^2}{2m}$$

Solving for p_z and plugging into

$$J = \oint p \, dq$$

we get

$$J_z = \sqrt{2m} \oint (E_z - mgz)^{1/2} \, dz$$

we can do this closed integral by integrating from 0 to h and multiplying by 2.

$$J_z = 2\sqrt{2m}\frac{2}{3}(E_z - mgz)^{3/2}(\frac{-1}{mg})\Big|_0^h$$

The original energy given to it in the z direction will be mgh, its potential energy when released from rest. Thus the first part of this evaluated integral is zero. Only the second part remains:

$$J_z = \frac{4}{3}\sqrt{2m}\frac{1}{mg}E_z^{3/2}$$

Solved in terms of E_z

$$E_z = (\frac{3}{4}g\sqrt{\frac{m}{2}}J_z)^{2/3}$$

The frequency is

$$\nu_z = \frac{\partial E_z}{\partial J_z} = (\frac{2}{3}(\frac{3}{4}g\sqrt{\frac{m}{2}})^{2/3})\frac{1}{J^{1/3}}$$

All we have to do now is plug what J_z is into this expression and simplify the algebra. As you may already see there are many different steps to take to simplify, I'll show one.

$$\nu_z = (\frac{2}{3}(\frac{3}{4}g\sqrt{\frac{m}{2}})^{2/3})\frac{1}{[\frac{4}{3g}\sqrt{\frac{2}{m}}(mgh)^{3/2}]^{1/3}}$$

Now we have a wonderful mess. Lets gather the numbers, and the constants to one side

$$\nu_z = \frac{\frac{2}{3}(\frac{3}{4})^{2/3}\frac{1}{2^{1/3}}}{(\frac{4}{3})^{1/3}2^{1/6}}\frac{g^{2/3}m^{1/3}g^{1/3}m^{1/6}}{m^{1/2}g^{1/2}h^{1/2}}$$

You may see, with some careful observation, that the m's cancel, and the constant part becomes

$$\frac{g^{1/2}}{h^{1/2}}$$

The number part simplifies down to

$$\frac{1}{2\sqrt{2}}$$

Thus we have

$$\nu_z = \frac{1}{2\sqrt{2}}\sqrt{\frac{g}{h}} = \frac{1}{2}\sqrt{\frac{g}{2h}}$$

as we were looking for from Newton's trivial method. Yay! Our two frequencies together

$$\nu_\theta = \frac{\dot{\theta}}{2\pi}$$

$$\nu_z = \frac{1}{2}\sqrt{\frac{g}{2h}}$$

The condition for the same path to be retraced is that the ratio of the frequencies to be a rational number. This is explained via closed Lissajous figures and two commensurate expressions at the bottom of page 462 in Goldstein.