Homework 4: # 2.18, 2.21, 3.13, 3.14, 3.20

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2.18 A point mass is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed ω. Obtain the Lagrange equations of motion assuming the only external forces arise from gravity. What are the constants of motion? Show that if ω is greater than a critical value ω_0, there can be a solution in which the particle remains stationary on the hoop at a point other than the bottom, but if $\omega < \omega_0$, the only stationary point for the particle is at the bottom of the hoop. What is the value of ω_0?

Answer:

To obtain the equations of motion, we need to find the Lagrangian. We only need one generalized coordinate, because the radius of the hoop is constant, and the point mass is constrained to this radius, while the angular velocity, w is constant as well.

$$L = \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2 \sin^2\theta) - mga\cos\theta$$

Where the kinetic energy is found by spherical symmetry, and the potential energy is considered negative at the bottom of the hoop, and zero where the vertical is at the center of the hoop. My θ is the angle from the z-axis, and a is the radius.
The equations of motion are then:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$

$$ma^2\ddot{\theta} = ma^2\omega^2\sin\theta\cos\theta + mga\sin\theta$$

We see that the Lagrangian does not explicitly depend on time therefore the energy function, h, is conserved (Goldstein page 61).

$$h = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L$$

$$h = \dot{\theta}ma^2\dot{\theta} - \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mga\cos\theta$$

This simplifies to:

$$h = \frac{1}{2}ma^2\dot{\theta} - (\frac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta)$$

Because the 'energy function' has an identical value to the Hamiltonian, the effective potential is the second term,

$$V_{eff} = mga\cos\theta - \frac{1}{2}ma^2\omega^2\sin^2\theta$$

The partial of V_{eff} with respect to θ set equal to zero should give us a stationary point.

$$\frac{\partial V_{eff}}{\partial\theta} = mga\sin\theta + ma^2\omega^2\sin\theta\cos\theta = 0$$

$$ma\sin\theta(g + a\omega^2\cos\theta) = 0$$

This yields three values for θ to obtain a stationary point,

$$\theta = 0 \quad \theta = \pi \quad \theta = \arccos(-\frac{g}{a\omega^2})$$

At the top, the bottom, and some angle that suggests a critical value of ω.

$$\omega_0 = \sqrt{\frac{g}{a}}$$

The top of the hoop is unstable, but at the bottom we have a different story. If I set $\omega = \omega_0$ and graph the potential, the only stable minimum is at $\theta = \pi$, the bottom. Therefore anything $\omega < \omega_0$, $\theta = \pi$ is stable, and is the only stationary point for the particle.

If we speed up this hoop, $\omega > \omega_0$, our angle

$$\theta = \arccos(-\frac{\omega_0^2}{\omega^2})$$

is stable and $\theta = \pi$ becomes unstable. So the point mass moves up the hoop, to a nice place where it is swung around and maintains a stationary orbit.

2.21 A carriage runs along rails on a rigid beam, as shown in the figure below. The carriage is attached to one end of a spring of equilibrium length r_0 and force constant k, whose other end is fixed on the beam. On the carriage, another set of rails is perpendicular to the first along which a particle of mass m moves, held by a spring fixed on the beam, of force constant k and zero equilibrium length. Beam, rails, springs, and carriage are assumed to have zero mass. The whole system is forced to move in a plane about the point of attachment of the first spring, with a constant angular speed ω. The length of the second spring is at all times considered small compared to r_0.

- What is the energy of the system? Is it conserved?
- Using generalized coordinates in the laboratory system, what is the Jacobi integral for the system? Is it conserved?
- In terms of the generalized coordinates relative to a system rotating with the angular speed ω, what is the Lagrangian? What is the Jacobi integral? Is it conserved? Discuss the relationship between the two Jacobi integrals.

Answer:

Energy of the system is found by the addition of kinetic and potential parts. The kinetic, in the lab frame, (x, y), using Cartesian coordinates is

$$T(x, y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Potential energy is harder to write in lab frame. In the rotating frame, the system looks stationary, and its potential energy is easy to write down. I'll use (r, l) to denote the rotating frame coordinates. The potential, in the rotating frame is

$$V(r, l) = \frac{1}{2}k(r^2 + l^2)$$

Where r is simply the distance stretched from equilibrium for the large spring. Since the small spring has zero equilbrium length, then the potential energy for it is just $\frac{1}{2}kl^2$.

The energy needs to be written down fully in one frame or the other, so I'll need a pair of transformation equations relating the two frames. That is, relating (x, y) to (r, l). Solving for them, by drawing a diagram, yields

$$x = (r_0 + r)\cos\omega t - l\sin\omega t$$

$$y = (r_0 + r)\sin\omega t + l\cos\omega t$$

Manipulating these so I may find $r(x, y)$ and $l(x, y)$ so as to write the stubborn potential energy in terms of the lab frame is done with some algebra.

Multiplying x by $\cos\omega t$ and y by $\sin\omega t$, adding the two equations and solving for r yields

$$r = x\cos\omega t + y\sin\omega t - r_0$$

Multiplying x by sin and y by cos, adding and solving for l yields

$$l = -x\sin\omega t + y\cos\omega t$$

Plugging these values into the potential energy to express it in terms of the lab frame leaves us with

$$E(x,y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k((x\cos\omega t + y\sin\omega t - r_0)^2 + (-x\sin\omega t + y\cos\omega t)^2)$$

This energy is explicitly dependent on time. Thus it is NOT conserved in the lab frame. $E(x,y)$ is not conserved.

In the rotating frame this may be a different story. To find $E(r,l)$ we are lucky to have an easy potential energy term, but now our kinetic energy is giving us problems. We need

$$E(r,l) = T(r,l) + \frac{1}{2}k(l^2 + r^2)$$

Where in the laboratory frame, $T(x,y) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$. Taking derivatives of x and y yield

$$\dot{x} = -\omega(r_0 + r)\sin\omega t + \dot{r}\cos\omega t - l\omega\cos\omega t - \dot{l}\sin\omega t$$
$$\dot{y} = \omega(r_0 + r)\cos\omega t + \dot{r}\sin\omega t - l\omega\sin\omega t - \dot{l}\cos\omega t$$

Squaring both and adding them yields

$$\dot{x}^2 + \dot{y}^2 = \omega^2(r_0 + r)^2\dot{r}^2 + l^2\omega^2 + \dot{l}^2 + C.T.$$

Where cross terms, C.T. are

$$C.T. = 2\omega(r_0 + r)\dot{l} - 2\dot{r}l\omega$$

For kinetic energy we know have

$$T(r,l) = \frac{1}{2}m(\omega^2(r_0 + r)^2\dot{r}^2 + l^2\omega^2 + \dot{l}^2 + 2\omega(r_0 + r)\dot{l} - 2\dot{r}l\omega)$$

Collecting terms

$$T(r,l) = \frac{1}{2}m(\omega^2(r_0 + r + \frac{\dot{l}}{\omega})^2 + (\dot{r} - l\omega)^2)$$

Thus

$$E(r,l) = \frac{1}{2}m(\omega^2(r_0 + r + \frac{\dot{l}}{\omega})^2 + (\dot{r} - l\omega)^2) + \frac{1}{2}k(l^2 + r^2)$$

This has no explicit time dependence, therefore energy in the rotating frame is conserved. $E(r,l)$ is conserved.

In the laboratory frame, the Lagrangian is just $T(x,y) - V(x,y)$.

$$L(x,y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x,y)$$

Where

$$V(x,y) = \frac{1}{2}k((x\cos\omega t + y\sin\omega t - r_0)^2 + (-x\sin\omega t + y\cos\omega t)^2)$$

The Jacobi integral, or energy function is

$$h = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

We have

$$h = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} - L(x,y)$$

$$h = \dot{x}m\dot{x} + \dot{y}m\dot{y} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + V(x,y)$$

Notice that $V(x,y)$ does not have any dependence on $\dot{x}$ or $\dot{y}$. Bringing it together

$$h = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k((x\cos\omega t + y\sin\omega t - r_0)^2 + (-x\sin\omega t + y\cos\omega t)^2)$$

This is equal to the energy.

$$h(x,y) = E(x,y)$$

Because it is dependent on time,

$$\frac{d}{dt}h = -\frac{\partial L}{\partial t} \neq 0$$

we know $h(x,y)$ is not conserved in the lab frame.

For the rotating frame, the Lagrangian is

$$L(r,l) = T(r,l) - \frac{1}{2}k(r^2 + l^2)$$

Where

$$T(r,l) = \frac{1}{2}m(\omega^2(r_0 + r + \frac{\dot{l}}{\omega})^2 + (\dot{r} - l\omega)^2)$$

The energy function, or Jacobi integral is

$$h(r,l) = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{l}\frac{\partial L}{\partial \dot{l}} - L(r,l)$$

$$h(r,l) = \dot{r}m(\dot{r} - l\omega) + \dot{l}m\omega(r_0 + r + \frac{\dot{l}}{\omega}) - L(r,l)$$

Collecting terms, with some heavy algebra

$$h = (\dot{r} - l\omega)(m\dot{r} - \frac{1}{2}m(\dot{r} - l\omega)) + (r_0 + r + \frac{\dot{l}}{\omega})(m\omega\dot{l} - \frac{1}{2}m\omega^2(r_0 + r + \frac{\dot{l}}{\omega})) + \frac{1}{2}k(l^2 + r^2)$$

$$h = (\dot{r} - l\omega)(\frac{m\dot{r}}{2} + \frac{1}{2}ml\omega) + (r_0 + r + \frac{\dot{l}}{\omega})(\frac{1}{2}m\omega\dot{l} - \frac{1}{2}m\omega^2(r_0 + r)) + \frac{1}{2}k(l^2 + r^2)$$

More algebraic manipulation in order to get terms that look like kinetic energy,

$$h = \frac{1}{2}m(\dot{r}^2 + \dot{l}^2) + \frac{1}{2}k(l^2 + r^2) + \frac{1}{2}[\dot{r}ml\omega - l\omega m\dot{r} - ml^2\omega^2 + (r_0 + r)m\omega\dot{l} - m\omega^2(r_0 + r)^2 - m\omega\dot{l}(r_0 + r)]$$

Yields

$$h(r,l) = \frac{1}{2}m(\dot{r}^2 + \dot{l}^2) + \frac{1}{2}k(l^2 + r^2) - \frac{1}{2}m\omega^2(l^2 + (r_0 + r)^2)$$

This has no time dependence, and this nice way of writing it reveals an energy term of rotation in the lab frame that can't be seen in the rotating frame. It is of the from $E = -\frac{1}{2}I\omega^2$.

$$\frac{d}{dt}h = -\frac{\partial L}{\partial t} = 0$$

We have $h(r,l)$ conserved in the rotating frame.

3.13

- Show that if a particle describes a circular orbit under the influence of an attractive central force directed toward a point on the circle, then the force varies as the inverse-fifth power of the distance.
- Show that for orbit described the total energy of the particle is zero.
- Find the period of the motion.
- Find $\dot{x}$, $\dot{y}$ and v as a function of angle around the circle and show that all three quantities are infinite as the particle goes through the center of force.

Answer:

Using the differential equation of the orbit, equation (3.34) in Goldstein,

$$\frac{d^2}{d\theta^2}u + u = -\frac{m}{l^2}\frac{d}{du}V(\frac{1}{u})$$

Where $r = 1/u$ and with the origin at a point on the circle, a triangle drawn with r being the distance the mass is away from the origin will reveal

$$r = 2R\cos\theta$$

$$u = \frac{1}{2R\cos\theta}$$

Plugging this in and taking the derivative twice,

$$\frac{d}{d\theta}u = \frac{1}{2R}[-\cos^{-2}\theta(-\sin\theta)] = \frac{\sin\theta}{2R\cos^2\theta}$$

The derivative of this is

$$\frac{d}{d\theta}\frac{\sin\theta}{2R\cos^2\theta} = \frac{1}{2R}[\sin\theta(-2\cos^{-3}\theta)(-\sin\theta) + \cos^{-2}\theta\cos\theta]$$

Thus

$$\frac{d^2}{d\theta^2}u = \frac{1}{2R}[\frac{2\sin^2\theta}{\cos^3\theta} + \frac{\cos^2\theta}{\cos^3\theta}] = \frac{1+\sin^2\theta}{2R\cos^3\theta}$$

$$\frac{d^2}{d\theta^2}u + u = \frac{1+\sin^2\theta}{2R\cos^3\theta} + \frac{\cos^2\theta}{2R\cos^3\theta} = \frac{2}{2R\cos^3\theta}$$

That is

$$\frac{8R^2}{8R^3\cos^3\theta} = 8R^2u^3$$

Solving for $V(\frac{1}{u})$ by integrating yields,

$$V(\frac{1}{u}) = -\frac{8R^2l^2}{4m}u^4$$

and we have

$$V(r) = -\frac{2l^2R^2}{mr^4}$$

with force equal to

$$f(r) = -\frac{d}{dr}V(r) = -\frac{8l^2R^2}{mr^5}$$

This force is inversely proportional to r^5.

Is the energy zero? Well, we know $V(r)$, lets find $T(r)$ and hope its the negative of $V(r)$.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Where

$$r = 2R\cos\theta \qquad \rightarrow \qquad \dot{r} = -2R\sin\theta\dot{\theta}$$

$$r^2 = 4R^2\cos^2\theta \qquad \dot{r}^2 = 4R^2\sin^2\dot{\theta}^2$$

So, plugging these in,

$$T = \frac{1}{2}m(4R^2\sin^2\theta\dot{\theta}^2 + 4R^2\cos^2\theta\dot{\theta}^2)$$

$$T = \frac{m4R\dot{\theta}^2}{2} = 2mR^2\dot{\theta}^2$$

Put this in terms of angular momentum, l,

$$l = mr^2\dot{\theta}$$

$$l^2 = m^2r^4\dot{\theta}^2$$

$$T = 2mR^2\dot{\theta}^2 \quad \rightarrow \quad T = \frac{2R^2l^2}{mr^4}$$

Which shows that

$$E = T + V = \frac{2R^2l^2}{mr^4} - \frac{2R^2l^2}{mr^4} = 0$$

the total energy is zero.

The period of the motion can be thought of in terms of θ as r spans from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$$P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{d\theta} d\theta$$

This is

$$P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\dot{\theta}}$$

Because $\dot{\theta} = l/mr^2$ in terms of angular momentum, we have

$$P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{mr^2}{l} d\theta$$

$$P = \frac{m}{l} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 d\theta$$

From above we have r^2

$$P = \frac{m}{l} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4R^2 \cos^2 \theta d\theta$$

$$P = \frac{4mR^2}{l} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta = \frac{4mR^2}{l} (\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}) = \frac{4mR^2}{l} (\frac{\pi}{4} + \frac{\pi}{4})$$

And finally,

$$P = \frac{2m\pi R^2}{l}$$

For $\dot{x}$, $\dot{y}$, and v as a function of angle, it can be shown that all three quantities are infinite as particle goes through the center of force. Remembering that $r = 2R\cos\theta$,

$$x = r\cos\theta = 2R\cos^2\theta$$

$$y = r\sin\theta = 2R\cos\theta\sin\theta = R\sin 2\theta$$

Finding their derivatives,

$$\dot{x} = -4R\cos\theta\sin\theta = -2R\dot{\theta}\sin 2\theta$$

$$\dot{y} = 2R\dot{\theta}\cos 2\theta$$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = 2R\dot{\theta}$$

What is $\dot{\theta}$? In terms of angular momentum we remember

$$l = mr^2\dot{\theta}$$

Plugging in our r, and solving for $\dot{\theta}$

$$\dot{\theta} = \frac{l}{4mR^2\cos^2\theta}$$

As we got closer to the origin, θ becomes close to $\pm\frac{\pi}{2}$.

$$\theta = \pm(\frac{\pi}{2} - \delta)$$

Note that as

$$\delta \to 0 \qquad \theta \to \pm\frac{\pi}{2} \qquad \dot{\theta} \to \infty$$

All $\dot{x}$, $\dot{y}$ and v are directly proportional to the $\dot{\theta}$ term. The $\dot{x}$ may be questionable at first because it has a $\sin 2\theta$ and when $\sin 2\theta \to 0$ as $\theta \to \pi/2$ we may be left with $\infty * 0$. But looking closely at $\dot{\theta}$ we can tell that

$$\dot{x} = \frac{-4Rl\cos\theta\sin\theta}{4mR^2\cos^2\theta} = -\frac{l}{mR}\tan\theta$$

$$\tan\theta \to \infty \quad as \quad \theta \to \pm\frac{\pi}{2}$$

2.14

- For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that perihelion distance of the parabola is one-half the radius of the circle.
- Prove that in the same central force as above, the speed of a particle at any point in a parabolic orbit is $\sqrt{2}$ times the speed in a circular orbit passing through the same point.

Answer:

Using the equation of orbit, Goldstein equation 3.55,

$$\frac{1}{r} = \frac{mk}{l^2}[1 + \epsilon\cos(\theta - \theta')]$$

we have for the circle, $\epsilon = 0$

$$\frac{1}{r_c} = \frac{mk}{l^2} \to r_c = \frac{l^2}{mk}$$

For the parabola, $\epsilon = 1$

$$\frac{1}{r_p} = \frac{mk}{l^2}(1+1) \to r_p = \frac{l^2}{2mk}$$

So

$$r_p = \frac{r_c}{2}$$

The speed of a particle in a circular orbit is

$$v_c^2 = r^2\dot{\theta}^2 \quad \rightarrow \quad v_c^2 = r^2(\frac{l^2}{m^2r^4}) \quad \rightarrow \quad v_c = \frac{l}{mr}$$

In terms of k, this is equal to

$$\frac{l}{mr} = \frac{\sqrt{mrk}}{mr} = \sqrt{\frac{k}{mr}}$$

The speed of a particle in a parabola can be found by

$$v_p^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\dot{r} = \frac{d}{dt}(\frac{l^2}{mk(1+\cos\theta)}) = \frac{l^2\dot{\theta}}{mk(1+\cos\theta)^2}\sin\theta$$

Solving for v_p,

$$v_p^2 = r^2\dot{\theta}^2(\frac{sin^2\theta}{(1+\cos\theta)^2} + 1)$$

$$v_p^2 = r^2\dot{\theta}^2(\frac{2+2\cos\theta}{(1+\cos\theta)^2})$$

$$v_p^2 = \frac{2r^2\dot{\theta}^2}{1+\cos\theta}$$

Using r for a parabola from Goldstein's (3.55), and not forgetting that $k = l^2/mr$,

$$r = \frac{l^2}{mk(1+\cos\theta)} \rightarrow \dot{\theta}^2 = \frac{l^2}{m^2r^4}$$

we have

$$v_p^2 = \frac{2r^2l^2mkr}{m^2r^4l^2} \rightarrow v_p^2 = \frac{2k}{mr}$$

For the speed of the parabola, we then have

$$v_p = \sqrt{2}\sqrt{\frac{k}{mr}}$$

Thus

$$v_p = \sqrt{2}v_c$$

20. A uniform distribution of dust in the solar system adds to the gravitational attraction of the Sun on a planet an additional force

$$F = -mCr$$

where m is the mas of the planet, C is a constant proportional to the gravitational constant and the density of the dust, and r is the radius vector from the Sun to the planet(both considered as points). This additional force is very small compared to the direct Sun-planet gravitational force.

- Calculate the period for a circular orbit of radius r_0 of the planet in this combined field.
- Calculate the period of radial oscillations for slight disturbances from the circular orbit.
- Show that nearly circular orbits can be approximated by a precessing ellipse and find the precession frequency. Is the precession in the same or opposite direction to the orbital angular velocity?

Answer:

The equation for period is

$$T = \frac{2\pi}{\dot{\theta}}$$

For a circular orbit,

$$\dot{\theta} = \frac{l}{mr^2}$$

Thus

$$T = \frac{2\pi mr^2}{l}$$

Goldstein's equation after (3.58):

$$\frac{k}{r_0^2} = \frac{l^2}{mr_0^3}$$

In our case, we have an added force due to the dust,

$$mCr_0 + \frac{k}{r_0^2} = \frac{l^2}{mr_0^3}$$

Solving for l yields

$$l = \sqrt{mr_0 k + m^2 C r_0^4}$$

Plugging this in to our period,

$$T = \frac{2\pi m r^2}{\sqrt{m r_0 k + m^2 C r_0^4}} \quad \rightarrow \quad T = \frac{2\pi}{\sqrt{\frac{k}{m r_0^3} + C}}$$

Here the orbital angular velocity is

$$\omega_{orb} = \sqrt{\frac{k}{m r^3} + C}$$

This is nice because if the dust was not there, we would have $C = 0$ and our period would be

$$T_0 = \frac{2\pi}{\sqrt{\frac{k}{m r_0^2}}} \quad \rightarrow \quad \omega_0 = \sqrt{\frac{k}{m r_0^2}}$$

which agrees with $l = m r_0^2 \omega_0$ and $l = \sqrt{mrk}$.

The period of radial oscillations for slight disturbances from the circular orbit can be calculated by finding β. β is the number of cycles of oscillation that the particle goes through in one complete orbit. Dividing our orbital period by β will give us the period of the oscillations.

$$T_{osc} = \frac{T}{\beta}$$

Equation (3.45) in Goldstein page 90, states that for small deviations from circularity conditions,

$$u \equiv \frac{1}{r} = u_0 + a \cos \beta\theta$$

Substitution of this into the force law gives equation (3.46)

$$\beta^2 = 3 + \frac{r}{f} \frac{df}{dr}\bigg|_{r=r_0}$$

Solve this with $f = mCr + k/r^2$

$$\frac{df}{dr} = -\frac{2k}{r^3} + mC$$

$$\beta^2 = 3 + r \frac{-\frac{2k}{r^3} + mC}{\frac{k}{r^2} + mCr}$$

$$\beta^2 = \frac{\frac{k}{r^2} + 4mCr}{\frac{k}{r^2} + mCr} \quad \rightarrow \quad \beta^2 = \frac{\frac{k}{mr^3} + 4C}{\frac{k}{mr^3} + C}$$

Now

$$T_{osc} = \frac{T}{\beta} \qquad \beta = \frac{\sqrt{\frac{k}{mr^3} + 4C}}{\sqrt{\frac{k}{mr^3} + C}}$$

Therefore, our period of radial oscillations is

$$T_{osc} = \frac{2\pi}{\sqrt{\frac{k}{mr^3} + 4C}}$$

Here

$$\omega_r = \sqrt{\frac{k}{mr^3} + 4C}$$

A nearly circular orbit can be approximated by a precessing ellipse. The equation for an elliptical orbit is

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0}$$

with $e << 1$, for a nearly circular orbit, a precessing ellipse will hug closely to the circle that would be made by $e = 0$.

To find the precession frequency, I'm going to subtract the orbital angular velocity from the radial angular velocity,

$$\omega_{prec} = \omega_r - \omega_{orb}$$

$$\omega_{prec} = \sqrt{\frac{k}{mr^3} + 4C} - \sqrt{\frac{k}{mr^3} + C}$$

Fixing this up so as to use the binomial expansion,

$$\omega_{prec} = \sqrt{\frac{k}{mr^3}}(\sqrt{1 + \frac{4Cmr^3}{k}} - \sqrt{1 + \frac{Cmr^3}{k}})$$

Using the binomial expansion,

$$\omega_{prec} = \sqrt{\frac{k}{mr^3}}[1 + \frac{2Cmr^3}{k} - (1 + \frac{Cmr^3}{2k})] = \sqrt{\frac{k}{mr^3}}[\frac{2Cmr^3}{k} - \frac{Cmr^3}{2k}]$$

$$\omega_{prec} = \sqrt{\frac{k}{mr^3}}\frac{4Cmr^3 - Cmr^3}{2k} = \frac{3Cmr^3}{2k}\sqrt{\frac{k}{mr^3}} = \frac{3C}{2}\sqrt{\frac{mr^3}{k}}$$

Therefore,

$$\omega_{prec} = \frac{3C}{2\omega_0} \quad \rightarrow \quad f_{prec} = \frac{3C}{4\pi\omega_0}$$

Because the radial oscillations take on a higher angular velocity than the orbital angular velocity, the orbit is very nearly circular but the radial extrema comes a tiny bit more than once per period. This means that the orbit precesses opposite the direction of the orbital motion.

Another way to do it, would be to find change in angle for every oscillation,

$$\Delta\theta = 2\pi - \frac{2\pi}{\beta}$$

Using the ratios,

$$T_{prec} = \frac{2\pi}{\Delta\theta} T_{osc}$$

With some mean algebra, the period of precession is

$$T_{prec} = \frac{4\pi}{1 - \frac{\frac{k}{mr^3}+C}{\frac{k}{mr^3}+4C}} \frac{1}{\sqrt{\frac{k}{mr^3} + 4c}}$$

$$T_{prec} = \frac{4\pi(\sqrt{\frac{k}{mr^3} + 4C})}{\frac{k}{mr^3} + 4C - (\frac{k}{mr^3} + C)} = \frac{4\pi}{3C}\sqrt{\frac{k}{mr^3} + 4C}$$

Because C is very small compared to k, the approximation holds,

$$T_{prec} \approx \frac{4\pi}{3C}\sqrt{\frac{k}{mr^3}}$$

Therefore,

$$T_{prec} = \frac{4\pi}{3C}\omega_0 \quad \rightarrow \quad f_{prec} = \frac{3C}{4\pi\omega_0} \quad \rightarrow \quad \omega_{prec} = \frac{3C}{2\omega_0}$$

where $\omega_0 = \sqrt{\frac{k}{mr^3}}$.