Homework 9: # 8.19, 8.24, 8.25

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8.19
The point of suspension of a simple pendulum of length l and mass m is constrained to move on a parabola $z = ax^2$ in the vertical plane. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. Obtain the Hamilton's equations of motion.

Answer:

Let

$$x' = x + l\sin\theta$$
$$z' = ax^2 - l\cos\theta$$

Then

$$T = \frac{1}{2}m(\dot{x}'^2 + \dot{z}'^2)$$
$$U = mgz'$$

Solving in terms of generalized coordinates, x and θ, our Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + 4a^2x^2\dot{x}^2 + 4ax\dot{x}l\dot{\theta}\sin\theta + l^2\dot{\theta}^2) - mg(ax^2 - l\cos\theta)$$

Using

$$L = L_0 + \frac{1}{2}\tilde{\dot{q}}T\dot{q}$$

where $\dot{q}$ and T are matrices. We can see

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}$$

$$T = \begin{pmatrix} m(1 + 4a^2x^2) & ml(\cos\theta + 2ax\sin\theta) \\ ml(\cos\theta + 2ax\sin\theta) & ml^2 \end{pmatrix}$$

with

$$L_0 = -mg(ax^2 - l\cos\theta)$$

The Hamilitonian is

$$H = \frac{1}{2}\tilde{p}T^{-1}p - L_0$$

Inverting T by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

with the algebra,

$$\frac{1}{ad-bc} = \frac{1}{m^2l^2(1+4ax^2) - m^2l^2(\cos\theta + 2ax\sin\theta)^2}$$

this is

$$= \frac{1}{m^2l^2(\sin^2\theta + 4ax^2 - 4ax\cos\theta\sin\theta - 4a^2x^2\sin^2\theta)}$$

$$= \frac{1}{m^2l^2(\sin^2\theta - 4ax\sin\theta\cos\theta + 4a^2x^2\cos^2\theta)}$$

which I'll introduce, for simplicity's sake, Y.

$$= \frac{1}{m^2l^2(\sin\theta - 2ax\cos\theta)^2} \equiv \frac{1}{m^2l^2Y}$$

So now we have

$$T^{-1} = \frac{1}{m^2l^2Y}\begin{pmatrix} ml^2 & -ml(\cos\theta + 2ax\sin\theta) \\ -ml(\cos\theta + 2ax\sin\theta) & m(1+4a^2x^2) \end{pmatrix}$$

$$T^{-1} = \frac{1}{mY}\begin{pmatrix} 1 & -(\cos\theta + 2ax\sin\theta)/l \\ -(\cos\theta + 2ax\sin\theta)/l & (1+4a^2x^2)/l^2 \end{pmatrix}$$

I want to introduce a new friend, lets call him J

$$J \equiv (\cos\theta + 2ax\sin\theta)$$

$$Y \equiv (\sin\theta - 2ax\cos\theta)^2$$

So,

$$T^{-1} = \frac{1}{mY}\begin{pmatrix} 1 & -J/l \\ -J/l & (1+4a^2x^2)/l^2 \end{pmatrix}$$

Proceed to derive the Hamiltonian,

$$H = \frac{1}{2}\tilde{p}T^{-1}p - L_0$$

we can go step by step,

$$T^{-1}p = \frac{1}{mY}\left(\begin{array}{cc} 1 & -J/l \\ -J/l & (1+4a^2x^2/l^2 \end{array}\right)\left(\begin{array}{c} p_x \\ p_\theta \end{array}\right) = \frac{1}{mY}\left(\begin{array}{c} p_x-(J/l)p_\theta \\ (-J/l)p_x+(1+4a^2x^2/l^2)p_\theta \end{array}\right)$$

and

$$\tilde{p}T^{-1}p = \frac{1}{mY}(p_x^2 - \frac{J}{l}p_\theta p_x - \frac{J}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2)$$

the full Hamiltonian is

$$H = \frac{1}{2mY}(p_x^2 - 2\frac{J}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2) + mg(ax^2 - l\cos\theta)$$

plugging in my Y and J

$$H = \frac{1}{2m(\sin\theta - 2ax\cos\theta)^2}(p_x^2 - 2\frac{\cos\theta + 2ax\sin\theta}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2) + mg(ax^2 - l\cos\theta)$$

Now to find the equations of motion. They are

$$\dot{x} = \frac{\partial H}{\partial p_x} \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} \quad \dot{p}_x = -\frac{\partial H}{\partial x} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}$$

The first two are easy, especially with my substitutions.

$$\dot{x} = \frac{1}{mY}[p_x - \frac{J}{l}p_\theta] = \frac{1}{m(\sin\theta - 2ax\cos\theta)^2}[p_x - \frac{\cos\theta + 2ax\sin\theta}{l}p_\theta]$$

$$\dot{\theta} = \frac{1}{mYl}[-Jp_x + \frac{1+4a^2x^2}{l}p_\theta] = \frac{1}{ml(\sin\theta - 2ax\cos\theta)^2}[-(\cos\theta + 2ax\sin\theta)p_x + \frac{1+4a^2x^2}{l}p_\theta]$$

But the next two are far more involved. I handled the partial with respect to x by taking the product rule between the two main pieces, the fraction out front, and mess inside the parenthesis that has p terms. I then broke each p term and began grouping them. Go slowly, and patiently. After taking the derivative before grouping, my $\dot{p}_x$ looked like this:

$$\dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial x} = \frac{1}{2m(\sin\theta - 2ax\cos\theta)^2}[\frac{-4a\sin\theta}{l}p_\theta p_x + \frac{8a^2x}{l^2}p_\theta^2]$$

$$-\frac{-2(-2a\cos\theta)}{2m(\sin\theta - 2ax\cos\theta)^3}[p_x^2 - 2\frac{\cos\theta + 2ax\sin\theta}{l}p_\theta p_x + \frac{1+4a^2x^2}{l^2}p_\theta^2] + 2mgax$$

Now start simplifying. Lets group the p terms.

$$\frac{4a(\cos\theta+2ax\sin\theta)}{2ml^2[\sin\theta-2ax\cos\theta]^3}p_\theta^2$$

$$\frac{4a\cos\theta}{2m[\sin\theta-2ax\cos\theta]^3}p_x^2$$

and the longest one..

$$\frac{2a}{lm[\sin\theta-2ax\cos\theta]^3}[\sin^2\theta-2-2ax\cos\theta\sin\theta]p_\theta p_x$$

Adding them all up yields, for $\dot{p}_x$:

$$-\frac{\partial H}{\partial x}=-\frac{2a}{m[\sin\theta-2ax\cos\theta]^3}[\cos\theta p_x^2+\frac{\cos\theta+2ax\sin\theta}{l^2}p_\theta^2-\frac{2-\sin^2\theta+2ax\sin\theta\cos\theta}{l}p_xp_\theta]-2mgax$$

Now for the next one, $\dot{p}_\theta$:

$$\dot{p}_\theta=-\frac{\partial H}{\partial\theta}$$

Taking the derivative, you get a monster, of course

$$\frac{\partial H}{\partial\theta}=\frac{1}{2m[\sin\theta-2ax\cos\theta]^2}[\frac{2\sin\theta}{l}-\frac{4ax\cos\theta}{l}]p_\theta p_x$$

$$+[p_x^2-2\frac{\cos\theta+2ax\sin\theta}{l}p_\theta p_x+\frac{1+4a^2x^2}{l^2}p_\theta^2][\frac{-2(\cos\theta+2ax\sin\theta)}{2m[\sin\theta-2ax\cos\theta]^3}+mgl\sin\theta$$

separating terms..

$$\frac{-(\cos\theta+2ax\sin\theta)}{m[\sin\theta-2ax\cos\theta]^3}p_x^2$$

$$\frac{\cos\theta+2ax\sin\theta}{m[\sin\theta-2ax\cos\theta]^3}\frac{1+4a^2x^2}{l^2}p_\theta^2$$

and the longest one...

$$[\frac{(\sin\theta-2ax\cos\theta)^2}{lm[\sin\theta-2ax\cos\theta]^3}+\frac{2(\cos\theta+2ax\sin\theta)^2}{lm[\sin\theta-2ax\cos\theta]^3}]p_\theta p_x$$

add them all up for the fourth equation of motion, $\dot{p}_\theta$

$$-\frac{\partial H}{\partial\theta}=\frac{1}{m[\sin\theta-2ax\cos\theta]^3}[(\cos\theta+2ax\sin\theta)(p_x^2+\frac{1+4a^2x^2}{l^2}p_\theta^2)$$

$$-\frac{[(\sin\theta-2ax\cos\theta)^2+2(\cos\theta+2ax\sin\theta)^2]}{l}p_\theta p_x$$

Together in all their glory:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{m[\sin\theta - 2ax\cos\theta]^3}[(\cos\theta + 2ax\sin\theta)(p_x^2 + \frac{1+4a^2x^2}{l^2}p_\theta^2)$$

$$-\frac{[(\sin\theta - 2ax\cos\theta)^2 + 2(\cos\theta + 2ax\sin\theta)^2]}{l}p_\theta p_x$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{2a}{m[\sin\theta - 2ax\cos\theta]^3}[\cos\theta p_x^2 + \frac{\cos\theta + 2ax\sin\theta}{l^2}p_\theta^2 - \frac{2-\sin^2\theta + 2ax\sin\theta\cos\theta}{l}p_x p_\theta] - 2mgax$$

$$\dot{x} = \frac{1}{m(\sin\theta - 2ax\cos\theta)^2}[p_x - \frac{\cos\theta + 2ax\sin\theta}{l}p_\theta]$$

$$\dot{\theta} = \frac{1}{ml(\sin\theta - 2ax\cos\theta)^2}[-(\cos\theta + 2ax\sin\theta)p_x + \frac{1+4a^2x^2}{l}p_\theta]$$

8.24
A uniform cylinder of radius a and density ρ is mounted so as to rotate freely around a vertical axis. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track along which a mass point m can slide without friction. Suppose a particle starts at rest at the top of the cylinder and slides down under the influence of gravity. Using any set of coordinates, arrive at a Hamiltonian for the combined system of particle and cylinder, and solve for the motion of the system.

Answer:

My generalized coordinates will be θ, the rotational angle of the particle with respect to the cylinder, and ϕ the rotational angle of the cylinder. The moment of inertia of a cylinder is

$$I = \frac{1}{2}Ma^2 = \frac{1}{2}\rho\pi h a^4$$

There are three forms of kinetic energy in the Lagrangian. The rotational energy of the cylinder, the rotational energy of the particle, and the translational kinetic energy of the particle. The only potential energy of the system will be the potential energy due to the height of the particle. The hardest part of this Lagrangian to understand is likely the translational energy due to the particle. The relationship between height and angle of rotational for a helix is

$$h = c\theta$$

Where c is the distance between the coils of the helix. MathWorld gives a treatment of this under helix. Understand that if the cylinder was not rotating then the rotational kinetic energy of the particle would merely be $\frac{m}{2}a^2\dot{\theta}^2$, but the rotation of the cylinder is adding an additional rotation to the particle's position. Lets write down the Lagrangian,

$$L = \frac{1}{2}I\dot{\phi}^2 + \frac{m}{2}[a^2(\dot{\theta}+\dot{\phi})^2 + c^2\dot{\theta}^2] + mgc\theta$$

This is

$$L = L_0 + \frac{1}{2}\tilde{\dot{q}}\,T\dot{q}$$

Solve for T.

$$T = \begin{pmatrix} ma^2+mc^2 & ma^2 \\ ma^2 & I+ma^2 \end{pmatrix} \quad \dot{q} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Using the same 2 by 2 inverse matrix form from the previous problem, we may solve for T^{-1}.

$$T^{-1} = \frac{1}{(ma^2+mc^2)(I+ma^2) - m^2a^4}\begin{pmatrix} I+ma^2 & -ma^2 \\ -ma^2 & m(a^2+c^2) \end{pmatrix}$$

Now we can find the Hamiltonian.

$$H = \frac{1}{2}\tilde{p}\,T^{-1}p - L_0$$

This is

$$H = \frac{p_\theta^2(I+ma^2) - 2ma^2p_\theta p_\phi + p_\phi^2 m(a^2+c^2)}{2[m(a^2+c^2)(I+ma^2) - m^2a^4]} - mgc\theta$$

From the equations of motion, we can solve for the motion of the system. (duh!) Here are the EOM:

$$-\frac{\partial H}{\partial \theta} = \dot{p}_\theta = mgc$$

$$-\frac{\partial H}{\partial \phi} = \dot{p}_\phi = 0$$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{(I+ma^2)p_\theta - ma^2p_\phi}{m(a^2+c^2)(I+ma^2) - m^2a^4}$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \frac{-ma^2p_\theta + p_\phi m(a^2+c^2)}{m(a^2+c^2)(I+ma^2) - m^2a^4}$$

To solve for the motion, lets use the boundary conditions. $\dot{\theta}(0) = \dot{\phi}(0) = 0$ leads to $p_\phi(0) = p_\theta(0) = 0$ leads to

$$p_\theta = mgct \quad p_\phi = 0$$

Pluggin and chuggin into $\dot{\theta}$ and $\dot{\phi}$ and integrating, yields the motion

$$\phi = \frac{-m^2a^2gct^2}{2[m(a^2+c^2)(I+ma^2)-m^2a^4]}$$

$$\theta = \frac{(I+ma^2)mgct^2}{2[m(a^2+c^2)(I+ma^2)-m^2a^4]}$$

If we plug in $I=\frac{1}{2}Ma^2$ where M is the mass of the cylinder, we obtain

$$\phi = \frac{-mgct^2}{2[mc^2+\frac{1}{2}M(a^2+c^2)]}$$

$$\theta = \frac{(m+\frac{1}{2}M)gct^2}{2[mc^2+\frac{1}{2}M(a^2+c^2)]}$$

8.25
Suppose that in the previous exercise the cylinder is constrained to rotate uniformly with angular frequency ω. Set up the Hamiltonian for the particle in an inertial system of coordinates and also in a system fixed in the rotating cylinder. Identify the physical nature of the Hamiltonian in each case and indicate whether or not the Hamiltonians are conserved.

Answer:

In the laboratory system, the particle moves through an angle $\psi=\theta+\phi$. The cylinder moves uniformly, $\phi=\omega t$, so the kinetic energy

$$T=\frac{1}{2}ma^2(\dot{\theta}+\dot{\phi})^2+\frac{1}{2}mc^2\dot{\theta}^2$$

may be expressed

$$T=\frac{1}{2}ma^2\dot{\psi}^2+\frac{1}{2}mc^2(\dot{\psi}-\omega)^2$$

The potential energy may be written

$$U=-mgc(\psi-\omega t)$$

So we have

$$L=\frac{1}{2}m(a^2\dot{\psi}^2+c^2(\dot{\psi}-\omega)^2)+mgc(\psi-\omega t)$$

$$\frac{\partial L}{\partial \dot{q}}=p=ma^2\dot{\psi}+mc^2(\dot{\psi}-\omega)$$

and with

$$H=\frac{1}{2}(\tilde{p}-a)T^{-1}(\tilde{p}-a)-L_0$$

we find T^{-1} from

$$L = \frac{1}{2}\tilde{\dot{q}}\, T^{-1}\dot{q} + \dot{q}a + L_0$$

We can see things better if we spread out L

$$L = \frac{1}{2}ma^2\dot{\psi}^2 + \frac{1}{2}mc^2\dot{\psi}^2 - mc^2\omega\dot{\psi} + \frac{1}{2}mc^2\omega^2 + mgc(\psi - \omega t)$$

so

$$L_0 = \frac{1}{2}mc^2\omega^2 + mgc(\psi - \omega t)$$

and

$$T = [ma^2 + mc^2]$$

$$T^{-1} = \frac{1}{m(a^2 + c^2)}$$

Therefore, for our Hamiltonian, we have

$$H_{lab} = \frac{(p - mc^2\omega)^2}{2m(a^2 + c^2)} - \frac{mc^2\omega^2}{2} - mgc(\psi - \omega t)$$

This is dependent on time, therefore it is not the total energy.

For the Hamiltonian in the rotating cylinder's frame, we express the movement in terms of the angle θ this is with respect to the cylinder.

$$\psi = \theta + \phi = \theta + \omega t$$
$$\dot{\psi} = \dot{\theta} + \dot{\phi} = \dot{\theta} + \omega$$

$$T = \frac{1}{2}ma^2(\dot{\theta} + \omega)^2 + \frac{1}{2}mc^2\dot{\theta}^2$$
$$U = -mgc\theta$$

$$L = \frac{1}{2}ma^2(\dot{\theta} + \omega)^2 + \frac{1}{2}mc^2\dot{\theta}^2 + mgc\theta$$
$$L = \frac{1}{2}\tilde{\dot{q}}\, T\dot{q} + \dot{q}a + L_0$$

Spread out L

$$L = \frac{1}{2}[ma^2 + mc^2]\dot{\theta}^2 + ma^2\dot{\theta}\omega + \frac{1}{2}ma^2\omega^2 + mgc\theta$$

It becomes clear that

$$T = [ma^2 + mc^2]$$

$$T^{-1} = \frac{1}{ma^2 + mc^2}$$

$$L_0 = \frac{1}{2}ma^2\omega + mgc\theta$$

Using again,

$$H = \frac{1}{2}(p - a)T^{-1}(p - a) - L_0$$

we may write

$$H = \frac{(p - ma^2\omega)^2}{2m(a^2 + c^2)} - \frac{1}{2}ma^2\omega - mgc\theta$$

This is not explicitly dependent on time, it is time-independent, thus conserved.