Homework 6: # 4.1, 4.2, 4.10, 4.14, 4.15

Michael Good

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> 4.1
> Prove that matrix multiplication is associative. Show that the product of two orthogonal matrices is also orthogonal.

Answer:

Matrix associativity means

$$A(BC) = (AB)C$$

The elements for any row i and column j, are

$$A(BC) = \sum_k A_{ik}(\sum_m B_{km}C_{mj})$$

$$(AB)C = \sum_m(\sum_k A_{ik}B_{km})C_{mj}$$

Both the elements are the same. They only differ in the order of addition. As long as the products are defined, and there are finite dimensions, matrix multiplication is associative.

Orthogonality may be defined by

$$\widetilde{A}A = I$$

The Pauli spin matrices, σ_x, and σ_z are both orthogonal.

$$\tilde{\sigma_x}\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\tilde{\sigma_z}\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

The product of these two:

$$\sigma_x\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv q$$

is also orthogonal:

$$\tilde{q}q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

More generally, if

$$\widetilde{A}A = 1 \qquad \widetilde{B}B = 1$$

then both A, and B are orthogonal. We can look at

$$\widetilde{AB}AB = \sum_k (\widetilde{AB})_{ik}(AB)_{kj} = \sum_k AB_{ki}AB_{kj} = \sum_{k,s,r} a_{ks}b_{si}a_{kr}b_{rj}$$

The elements are

$$\sum_{k,s,r} a_{ks}b_{si}a_{kr}b_{rj} = \sum b_{si}a_{ks}a_{kr}b_{rj} = \sum b_{si}(\tilde{A}A)_{sr}b_{rj}$$

This is

$$\widetilde{AB}AB = \sum b_{si}\delta_{sr}b_{rj} = \tilde{B}B_{ij} = \delta_{ij}$$

Therefore the whole matrix is I and the product

$$\widetilde{AB}AB = I$$

is orthogonal.

4.2
Prove the following properties of the transposed and adjoint matrices:

$$\widetilde{AB} = \widetilde{B}\widetilde{A}$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

Answer:

For transposed matrices

$$\widetilde{AB} = \widetilde{AB}_{ij} = AB_{ji} = \sum a_{js}b_{si} = \sum b_{si}a_{js} = \sum \widetilde{B}_{is}\widetilde{A}_{sj} = (\widetilde{B}\widetilde{A})_{ij} = \widetilde{B}\widetilde{A}$$

As for the complex conjugate,

$$(AB)^\dagger = (\widetilde{AB})^*$$

From our above answer for transposed matrices we can say

$$\widetilde{AB} = \widetilde{B}\widetilde{A}$$

And so we have

$$(AB)^\dagger = (\widetilde{AB})^* = (\widetilde{B}\widetilde{A})^* = \widetilde{B}^*\widetilde{A}^* = B^\dagger A^\dagger$$

4.10
If B is a square matrix and A is the exponential of B, defined by the infinite series expansion of the exponential,

$$A \equiv e^B = 1 + B + \frac{1}{2}B^2 + ... + \frac{B^n}{n!} + ...,$$

then prove the following properties:

- $e^B e^C = e^{B+C}$, providing B and C commute.
- $A^{-1} = e^{-B}$
- $e^{CBC^{-1}} = CAC^{-1}$
- A is orthogonal if B is antisymmetric

Answer:

Providing that B and C commute;

$$BC - CB = 0 \qquad BC = CB$$

we can get an idea of what happens:

$$(1+B+\frac{B^2}{2}+O(B^3))(1+C+\frac{C^2}{2}+O(C^3)) = 1+C+\frac{C^2}{2}+B+BC+\frac{B^2}{2}+O(3)$$

This is

$$1+(B+C)+\frac{1}{2}(C^2+2BC+B^2)+O(3) = 1+(B+C)+\frac{(B+C)^2}{2}+O(3) = e^{B+C}$$

Because $BC = CB$ and where $O(3)$ are higher order terms with products of 3 or more matrices. Looking at the kth order terms, we can provide a rigorous proof.

Expanding the left hand side of

$$e^B e^C = e^{B+C}$$

and looking at the kth order term, by using the expansion for exp we get, noting that $i + j = k$

$$\sum_0^k \frac{B^i C^j}{i!j!} = \sum_0^k \frac{B^{k-j}C^j}{(k-j)!j!}$$

and using the binomial expansion on the right hand side for the kth order term, (a proof of which is given in Riley, Hobsen, Bence):

$$\frac{(B+C)^k}{k!} = \frac{1}{k!}\sum_0^k \frac{k!}{(k-j)!j!}B^{k-j}C^j = \sum_0^k \frac{B^{k-j}C^j}{(k-j)!j!}$$

we get the same term. QED.
To prove

$$A^{-1} = e^{-B}$$

We remember that

$$A^{-1}A = 1$$

and throw e^{-B} on the right

$$A^{-1}Ae^{-B} = 1e^{-B}$$

$$A^{-1}e^{B}e^{-B} = e^{-B}$$

and from our above proof we know $e^B e^C = e^{B+C}$ so

$$A^{-1}e^{B-B} = e^{-B}$$

Presto,

$$A^{-1} = e^{-B}$$

To prove

$$e^{CBC^{-1}} = CAC^{-1}$$

its best to expand the *exp*

$$\sum_0^\infty \frac{1}{n!}(CBC^{-1})^n = 1+CBC^{-1}+\frac{CBC^{-1}CBC^{-1}}{2}+....+\frac{CBC^{-1}CBC^{-1}CBC^{-1}...}{n!}+...$$

Do you see how the middle $C^{-1}C$ terms cancel out? And how they cancel each out n times? So we are left with just the C and C^{-1} on the outside of the B's.

$$\sum_0^\infty \frac{1}{n!}(CBC^{-1})^n = \sum_0^\infty \frac{1}{n!}CB^nC^{-1} = Ce^BC^{-1}$$

Remember $A = e^B$ and we therefore have

$$e^{CBC^{-1}} = CAC^{-1}$$

To prove A is orthogonal

$$\widetilde{A} = A^{-1}$$

if B is antisymmetric

$$-B = \widetilde{B}$$

We can look at the transpose of A

$$\widetilde{A} = \widetilde{\sum_0^\infty \frac{B^n}{n!}} = \sum_0^\infty \frac{\widetilde{B}^n}{n!} = \sum_0^\infty \frac{(-B)^n}{n!} = e^{-B}$$

But from our second proof, we know that $e^{-B} = A^{-1}$, so

$$\widetilde{A} = A^{-1}$$

and we can happily say A is orthogonal.

4.14

- Verify that the permutation symbol satisfies the following identity in terms of Kronecker delta symbols:

$$\epsilon_{ijp}\epsilon_{rmp} = \delta_{ir}\delta_{jm} - \delta_{im}\delta_{jr}$$

- Show that

$$\epsilon_{ijp}\epsilon_{ijk} = 2\delta_{pk}$$

Answer:
To verify this first identity, all we have to do is look at the two sides of the equation, analyzing the possibilities, i.e. if the right hand side has

$$i = r \quad j = m \neq i$$

we get $+1$. If

$$i = m \quad j = r \neq i$$

we get -1. For any other set of i, j, r, and m we get 0.

For the left hand side, lets match conditions, if

$$i = r \quad j = m \neq i$$

then $\epsilon_{ijp} = \epsilon_{rmp}$ and whether or not ϵ_{ijp} is ± 1 the product of the two gives a $+1$. If

$$i = m \quad j = r \neq i$$

then $\epsilon_{rmp} = \epsilon_{jip} = -\epsilon_{ijp}$ and whether or not ϵ_{ijp} is ± 1 the product is now equal to -1.

These are the only nonzero values because for i, j, r, m, none can have the same value as p. Since there are only three values, that any of the subscripts may take, the only non-zero values are the ones above. (not all four subscripts may be equal because then it would be $\epsilon = 0$ as if $i = j$ or $r = m$).

To show that

$$\epsilon_{ijp}\epsilon_{ijk} = 2\delta_{pk}$$

we can use our previous identity, cast in a different form:

$$\epsilon_{ijk}\epsilon_{imp} = \delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km}$$

This is equivalent because the product of two Levi-Civita symbols is found from the deteriment of a matrix of delta's, that is

$$\epsilon_{ijk}\epsilon_{rmp} = \delta_{ir}\delta_{jm}\delta_{kp} + \delta_{im}\delta_{jp}\delta_{kr} + \delta_{ip}\delta_{jr}\delta_{km} - \delta_{im}\delta_{jr}\delta_{kp} - \delta_{ir}\delta_{jp}\delta_{km} - \delta_{ip}\delta_{jm}\delta_{kr}$$

For our different form, we set $i = r$. If we also set $j = m$, this is called 'contracting' we get

$$\epsilon_{ijk}\epsilon_{ijp} = \delta_{jj}\delta_{kp} - \delta_{jp}\delta_{kj}$$

Using the summation convention, $\delta_{jj} = 3$,

$$\epsilon_{ijk}\epsilon_{ijp} = 3\delta_{kp} - \delta_{kp}$$

$$\epsilon_{ijk}\epsilon_{ijp} = 2\delta_{kp}$$

4.15
Show that the components of the angular velocity along the space set of axes are given in terms of the Euler angles by

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$

Answer:

Using the same analysis that Goldstein gives to find the angular velocity along the body axes (x', y', z') we can find the angular velocity along the space axes (x, y, z). To make a drawing easier, its helpful to label the axes of rotation for $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$.

$$\dot{\theta} \to L.O.N.$$
$$\dot{\psi} \to z'$$
$$\dot{\phi} \to z$$

We want

$$\omega_x = \dot{\theta}_x + \dot{\psi}_x + \dot{\phi}_x$$
$$\omega_y = \dot{\theta}_y + \dot{\psi}_y + \dot{\phi}_y$$
$$\omega_z = \dot{\theta}_z + \dot{\psi}_z + \dot{\phi}_z$$

Lets start with ω_z first to be different. If we look at the diagram carefully on page 152, we can see that $\dot{\theta}$ is along the line of nodes, that is θ revolves around the line of nodes. Therefore because the line of nodes is perpendicular to the z space axis there is no component of θ contributing to angular velocity around the z space axis. $\dot{\theta}_z = 0$. What about $\dot{\psi}_z$? Well, ψ revolves around z'. So there is a component along z due to a changing ψ. That component depends on how much angle there is between z' and z, which is θ. Does this makes sense? We find the z part, which is the adjacent side to θ. Thus we have $\dot{\psi}_z = \dot{\psi}\cos\theta$. Now lets look at $\dot{\phi}_z$. We can see that ϕ just revolves around z in the first place! Right? So there is no need to make any 'transformation' or make any changes. Lets take $\dot{\phi}_z = \dot{\phi}$. Add them all up for our total ω_z.

$$\omega_z = \dot{\theta}_z + \dot{\psi}_z + \dot{\phi}_z = 0 + \dot{\psi}\cos\theta + \dot{\phi}$$

Now lets do the harder ones. Try ω_x. What is $\dot{\theta}_x$? Well, $\dot{\theta}$ is along the line of nodes, that is, θ changes and revolves around the line of nodes axis. To find the x component of that, we just see that the angle between the line of nodes and the x axis is only ϕ, because they both lie in the same xy plane. Yes? So $\dot{\theta}_x = \dot{\theta}\cos\phi$. The adjacent side to ϕ with $\dot{\theta}$ as the hypotenuse. Lets look at $\dot{\phi}_x$. See how ϕ revolves around the z axis? Well, the z axis is perpendicular to the x axis there for there is no component of $\dot{\phi}$ that contributes to the x space axis. $\dot{\phi}_x = 0$. Now look at $\dot{\psi}_x$. We can see that $\dot{\psi}$ is along the z' body axis, that is, it is in a whole different plane than x. We first have to find the component in the same xy plane, then find the component of the x direction. So to get into the xy plane we can take $\dot{\psi}_{x,y} = \dot{\psi}\sin\theta$. Now its in the same plane. But where is it facing in this plane? We can see that depends on the angle ϕ. If $\phi = 0$ we would have projected it right on top of the $y - axis$! So we can make sure that if $\phi = 0$ we have a zero component for x by multiplying by $\sin\phi$. So we get after two projections, $\dot{\psi}_x = \dot{\psi}\sin\theta\sin\phi$. Add these all up for our total ω_x, angular velocity in the x space axis.

$$\omega_x = \dot{\theta}_x + \dot{\psi}_x + \dot{\phi}_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi + 0$$

I'll explain ω_y for kicks, even though the process is exactly the same. Look for $\dot{\theta}_y$. $\dot{\theta}$ is along the line of nodes. Its y component depends on the angle ϕ. So project it to the y axis. $\dot{\theta}_y = \dot{\theta}\sin\phi$. Look for $\dot{\psi}_y$. Its in a different plane again, so two projections are necessary to find its component. Project down to the xy plane like we did before, $\dot{\psi}_{x,y} = \dot{\psi}\sin\theta$ and now we remember that if $\phi = 0$ we would have exactly placed it on top of the y axis. Thats good! So lets make it if $\phi = 0$ we have the full $\dot{\psi}\sin\theta$, (ie multiply by $\cos\phi$ because $\cos 0 = 1$). But we also have projected it in the opposite direction of the positive y direction, (throw in a negative). So we have $\dot{\psi}_y = -\dot{\psi}\sin\theta\cos\phi$. For $\dot{\phi}_y$ we note that ϕ revolves around the z axis, completely perpendicular to y. Therefore no component in the y direction. $\dot{\phi}_y = 0$. Add them all up

$$\omega_y = \dot{\theta}_y + \dot{\psi}_y + \dot{\phi}_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi + 0$$

Here is all the ω's together

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$$