Homework 1: # 1, 2, 6, 8, 14, 20

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1. Show that for a single particle with constant mass the equation of motion implies the follwing differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

Answer:

$$\frac{dT}{dt} = \frac{d(\frac{1}{2}mv^2)}{dt} = m\mathbf{v} \cdot \dot{\mathbf{v}} = m\mathbf{a} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v}$$

with time variable mass,

$$\frac{d(mT)}{dt} = \frac{d}{dt}(\frac{p^2}{2}) = \mathbf{p} \cdot \dot{\mathbf{p}} = \mathbf{F} \cdot \mathbf{p}.$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation:

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2.$$

Answer:

$$M\mathbf{R} = \sum m_i \mathbf{r}_i$$

$$M^2\mathbf{R}^2 = \sum_{i,j} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j$$

Solving for $\mathbf{r}_i \cdot \mathbf{r}_j$ realize that $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. Square $\mathbf{r}_i - \mathbf{r}_j$ and you get

$$r_{ij}^2 = r_i^2 - 2\mathbf{r}_i \cdot \mathbf{r}_j + r_j^2$$

Plug in for $\mathbf{r}_i \cdot \mathbf{r}_j$

$$\mathbf{r}_i \cdot \mathbf{r}_j = \frac{1}{2}(r_i^2 + r_j^2 - r_{ij}^2)$$

$$M^2R^2 = \frac{1}{2}\sum_{i,j} m_i m_j r_i^2 + \frac{1}{2}\sum_{i,j} m_i m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = \frac{1}{2}M\sum_{i} m_i r_i^2 + \frac{1}{2}M\sum_{j} m_j r_j^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

$$M^2R^2 = M\sum_{i} m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

6. A particle moves in the xy plane under the constraint that its velocity vector is always directed toward a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

Answer:

The abscissa is the x-axis distance from the origin to the point on the x-axis that the velocity vector is aimed at. It has the distance $f(t)$.

I claim that the ratio of the velocity vector components must be equal to the ratio of the vector components of the vector that connects the particle to the point on the x-axis. The directions are the same. The velocity vector components are:

$$v_y = \frac{dy}{dt}$$

$$v_x = \frac{dx}{dt}$$

The vector components of the vector that connects the particle to the point on the x-axis are:

$$V_y = y(t)$$

$$V_x = x(t) - f(t)$$

For these to be the same, then

$$\frac{v_y}{v_x} = \frac{V_y}{V_x}$$

$$\frac{dy}{dx} = \frac{y(t)}{x(t) - f(t)}$$

$$\frac{dy}{y(t)} = \frac{dx}{x(t) - f(t)}$$

This cannot be integrated with $f(t)$ being arbitrary. Thus the constraint is nonholonomic. If the constraint was holonomic then

$$F(x, y, t) = 0$$

would be true. If an arbitrary, but small change of dx, dy, dt was made subject to the constraint then the equation

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial t} dt = 0$$

would hold. From this it can be seen our constraint equation is actually

$$ydx + (f(t) - x)dy + (0)dt = 0$$

Thus

$$\frac{\partial F}{dt} = 0 \qquad \frac{\partial F}{\partial x} = yI \qquad \frac{\partial F}{\partial y} = (f(t) - x)I$$

where I is our integrating factor, $I(x, y, t)$. The first equation shows $F = F(x, y)$ and the second equation that $I = I(x, y)$. The third equation shows us that all of this is impossible because

$$f(t) = \frac{\partial F}{\partial y} \frac{1}{I(x, y)} + x$$

where $f(t)$ is only dependent on time, but the right side depends only on x and y. There can be no integrating factor for the constraint equation and thus it means this constraint is nonholonomic.

8. If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, ..., q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

Answer:

Let's directly substitute L' into Lagrange's equations.

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} - \frac{\partial L'}{\partial q} = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}(L + \frac{dF}{dt}) - \frac{\partial}{\partial q}(L + \frac{dF}{dt}) = 0$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\frac{dF}{dt}] - \frac{\partial L}{\partial q} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

On the left we recognized Lagrange's equations, which we know equal zero. Now to show the terms with F vanish.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial \dot{F}}{\partial q}$$

This is shown to be true because

$$\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$

and

$$\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial F}{\partial q}$$

$$= \frac{\partial}{\partial t}\frac{\partial F}{\partial q} + \frac{\partial}{\partial q}\frac{\partial F}{\partial q}\dot{q}$$

$$= \frac{\partial}{\partial q}[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}] = \frac{\partial \dot{F}}{\partial q}$$

Thus as Goldstein reminded us, $L = T - V$ is a suitable Lagrangian, but it is not the only Lagrangian for a given system.

14. Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

Answer:

$$T = T_1 + T_2$$

Where T_1 equals the kinetic energy of the center of mass, and T_2 is the kinetic energy about the center of mass. I will keep these two parts separate.

Solve for T_1 first, its the easiest:

$$T_1 = \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}(2m)(a\dot{\psi})^2 = ma^2\dot{\psi}^2$$

Solve for T_2, realizing that the rigid rod is probably not restricted to just the X-Y plane. The Z-axis adds more complexity to the problem.

$$T_2 = \frac{1}{2}Mv^2 = mv^2$$

Solve for v^2 about the center of mass. The angle ϕ will be the angle in the x-y plane, while the angle θ will be the angle from the z-axis.

If $\theta = 90^o$ and $\phi = 0^o$ then $x = l/2$ so:

$$x = \frac{l}{2}\sin\theta\cos\phi$$

If $\theta = 90^o$ and $\phi = 90^o$ then $y = l/2$ so:

$$y = \frac{l}{2}\sin\theta\sin\phi$$

If $\theta = 0^o$, then $z = l/2$ so:

$$z = \frac{l}{2}\cos\theta$$

Find v^2:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$$

$$\dot{x} = \frac{l}{2}(\cos\phi\cos\theta\dot{\theta} - \sin\theta\sin\phi\dot{\phi})$$

$$\dot{y} = \frac{1}{2}(\sin\phi\cos\theta\dot{\theta} + \sin\theta\cos\phi\dot{\phi})$$

$$\dot{z} = -\frac{l}{2}\sin\theta\dot{\theta}$$

Carefully square each:

$$\dot{x}^2 = \frac{l^2}{4}\cos^2\phi\cos^2\theta\dot{\theta}^2 - 2\frac{l}{2}\sin\theta\sin\phi\dot{\phi}\frac{l}{2}\cos\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\sin^2\phi\dot{\phi}^2$$

$$\dot{y}^2 = \frac{l^2}{4}\sin^2\phi\cos^2\theta\dot{\theta}^2 + 2\frac{l}{2}\sin\theta\cos\phi\dot{\phi}\frac{l}{2}\sin\phi\cos\theta\dot{\theta} + \frac{l^2}{4}\sin^2\theta\cos^2\phi\dot{\phi}^2$$

$$\dot{z}^2 = \frac{l^2}{4}\sin^2\theta\dot{\theta}^2$$

Now add, striking out the middle terms:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \frac{l^2}{4}[\cos^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\sin^2\phi\dot{\phi}^2+\sin^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\cos^2\phi\dot{\phi}^2+\sin^2\theta\dot{\theta}^2]$$

Pull the first and third terms inside the brackets together, and pull the second and fourth terms together as well:

$$v^2 = \frac{l^2}{4}[\cos^2\theta\dot{\theta}^2(\cos^2\phi+\sin^2\phi)+\sin^2\theta\dot{\phi}^2(\sin^2\phi+\cos^2\phi)+\sin^2\theta\dot{\theta}^2]$$

$$v^2 = \frac{l^2}{4}(\cos^2\theta\dot{\theta}^2+\sin^2\theta\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

$$v^2 = \frac{l^2}{4}(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

Now that we finally have v^2 we can plug this into T_2

$$T = T_1 + T_2 = ma^2\dot{\psi}^2 + m\frac{l^2}{4}(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

I want to emphasize again that T_1 is the kinetic energy of the total mass around the center of the circle while T_2 is the kinetic energy of the masses about the center of mass.

20. A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V_2(x)$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this system.

Answer:

Correcting for error,

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x)$$

Finding the equations of motion from Euler-Lagrange formulation:

$$\frac{\partial L}{\partial x} = +m\dot{x}^2V'(x) - 2V(x)V'(x)$$

$$\frac{\partial L}{\partial \dot{x}} = +\frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m^2\dot{x}^2\ddot{x} + 2mV(x)\ddot{x} + 2m\dot{x}V'(x)\dot{x}$$

Thus

$$-m\dot{x}^2V' + 2VV' + m^2\dot{x}^2\ddot{x} + 2mV\ddot{x} + 2m\dot{x}V'(x)\dot{x} = 0$$

$$m\dot{x}^2V' + 2VV' + m^2\dot{x}^2\ddot{x} + 2mV\ddot{x} = 0$$

is our equation of motion. But we want to interpret it. So lets make it look like it has useful terms in it, like kinetic energy and force. This can be done by dividing by 2 and separating out $\frac{1}{2}mv^2$ and ma's.

$$\frac{m\dot{x}^2}{2}V' + VV' + \frac{m\dot{x}^2}{2}m\ddot{x} + m\ddot{x}V = 0$$

Pull V' terms together and $m\ddot{x}$ terms together:

$$(\frac{m\dot{x}^2}{2} + V)V' + m\ddot{x}(\frac{m\dot{x}^2}{2} + V) = 0$$

Therefore:

$$(\frac{m\dot{x}^2}{2} + V)(m\ddot{x} + V') = 0$$

Now this looks like $E \cdot E' = 0$ because $E = \frac{m\dot{x}^2}{2} + V(x)$. That would mean

$$\frac{d}{dt}E^2 = 2EE' = 0$$

Which reveals that E^2 is a constant. If we look at $t = 0$ and the starting energy of the particle, then we will notice that if $E = 0$ at $t = 0$ then $E = 0$ for all other times. If $E \neq 0$ at $t = 0$ then $E \neq 0$ all other times while $m\ddot{x} + V' = 0$.