Homework 1: # 1.21, 2.7, 2.12

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1.21. Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until m_1 reaches the hole.)

Answer:

The generalized coordinates for the system are θ, the angle m_1 moves round on the table, and r the length of the string from the hole to m_1. The whole motion of the system can be described by just these coordinates. To write the Lagrangian, we will want the kinetic and potential energies.

$$T = \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -m_2 g(R - r)$$

The kinetic energy is just the addition of both masses, while V is obtained so that $V = -mgR$ when $r = 0$ and so that $V = 0$ when $r = R$.

$$L = T - V = \frac{1}{2}(m_2 + m_1)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 + m_2 g(R - r)$$

To find the Lagrangian equations or equations of motion, solve for each component:

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(m_1 r^2\dot{\theta}) = m_1 r^2\ddot{\theta} + 2m_1 r\dot{r}\dot{\theta}$$

Thus

$$m_1 r(r\ddot{\theta} + 2\dot{\theta}\dot{r}) = 0$$

and

$$\frac{\partial L}{\partial r} = -m_2 g + m_1 r\dot{\theta}^2$$

$$\frac{\partial L}{\partial \dot{r}} = (m_2 + m_1)\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = (m_2 + m_1)\ddot{r}$$

Thus

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

Therefore our equations of motion are:

$$\frac{d}{dt}(m_1 r^2\dot{\theta}) = m_1 r(r\ddot{\theta} + 2\dot{\theta}\dot{r}) = 0$$

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

See that $m_1 r^2\dot{\theta}$ is constant, because $\frac{d}{dt}(m_1 r^2\dot{\theta}) = 0$. It is angular momentum. Now the Lagrangian can be put in terms of angular momentum and will lend the problem to interpretation. We have $\dot{\theta} = l/m_1 r^2$, where l is angular momentum. The equation of motion

$$m_2 g - m_1 r\dot{\theta}^2 + (m_2 + m_1)\ddot{r} = 0$$

becomes

$$(m_1 + m_2)\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0$$

The problem has been reduced to a single non-linear second-order differential equation. The next step is a nice one to notice. If you take the first integral you get

$$\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} + m_2 g r + C = 0$$

To see this, check by assuming that $C = -m_2 g R$:

$$\frac{d}{dt}(\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} - m_2 g(R - r)) = (m_1 + m_2)\dot{r}\ddot{r} - \frac{l^2}{m_1 r^3}\dot{r} + m_2 g\dot{r} = 0$$

$$(m_1 + m_2)\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0$$

Because this term is T *plus* V, this is the total energy, and because its time derivative is constant, energy is conserved.

2.7 In Example 2 of Section 2.1 we considered the problem of the minimum surface of revolution. Examine the symmetric case $x_1 = x_2$, $y_2 = -y_1 > 0$, and express the condition for the parameter a as a transcendental equation in terms of the dimensionless quantities $k = x_2/a$, and $\alpha = y_2/x_2$. Show that for α greater than a certain value α_0 two values of k are possible, for $\alpha = \alpha_0$ only one value of k is possible, while if $\alpha < \alpha_0$ no real value of k (or a) can be found, so that no catenary solution exists in this region. Find the value of α_0, numerically if necessary.

Answer:

Starting with Goldstein's form for a catenary, in section 2.2, not 2.1,

$$x = a \cosh \frac{y-b}{a}$$

and recognizing by symmetry that the soap film problem and the catenary problem are the same. In Marion and Thorton this is made clear (pg 222). Also, in a similar way to MathWorld's analysis of a surface of revolution, it is clear that y and x, when interchanged, change the shape of the catenary to be about the x-axis.

$$y = a \cosh \frac{x-b}{a}$$

To preserve symmetry, $x_1 = -x_2$ and $y_2 = y_1$. This switch makes sense because if you hang a rope from two points, its going to hang between the points with a droopy curve, and fall straight down after the points. This shaped revolved around the x-axis looks like a horizontal worm hole. This is the classic catenary curve, or catenoid shape. The two shapes are physically equivalent, and take on different mathematical forms. With this, we see that

$$y_1 = a \cosh \frac{x_1 - b}{a} \qquad y_2 = a \cosh \frac{x_2 - b}{a}$$

holds. The two endpoints are (x_0, y_0) and $(-x_0, y_0)$.

Thus

$$y_0 = a \cosh \frac{x_0 - b}{a} = a \cosh \frac{-x_0 - b}{a}$$

and because

$$\cosh(-x) = \cosh(x)$$

we have

$$\cosh \frac{-x_0 + b}{a} = \cosh \frac{-x_0 - b}{a}$$

$$-x_0 + b = -x_0 - b$$

$$b = -b \qquad b = 0$$

By symmetry, with the center of the shape or rings at the origin, $b = 0$, simplifies the problem to a much nicer form:

$$y = a \cosh \frac{x}{a}$$

Including our end points:

$$y_0 = a \cosh \frac{x_0}{a}$$

In terms of the dimensionless quantities,

$$\rho = x_2/a \qquad \beta = \frac{1}{\alpha} = y_2/x_2$$

,

the equation is

$$y_0 = a \cosh \frac{x_0}{a}$$

$$\frac{y_0}{x_0} = \frac{a \cosh \rho}{x_0}$$

$$\frac{y_0}{x_0} = \frac{\cosh(\rho)}{\rho}$$

$$\beta = \frac{\cosh(\rho)}{\rho}$$

The minimum value of β in terms of ρ can be found by taking the derivative, and setting to zero:

$$0 = \frac{d}{d\rho}(\frac{1}{\rho} \cosh \rho) = \frac{1}{\rho} \sinh \rho - \frac{\cosh \rho}{\rho^2}$$

$$\sinh \rho = \frac{\cosh \rho}{\rho}$$

$$\rho = \coth \rho$$

Thus, solved numerically, $\rho \approx 1.2$. Plugging this in to find β_0, the value is:

$$\beta_0 \approx 1.51$$

Since

$$\beta_0 = \frac{1}{\alpha_0}$$

,

$$\alpha_0 \approx .66$$

This symmetric but physically equivalent example is not what the problem asked for, but I think its interesting. If we start at Goldstein's equation, again, only this time recognize $b = 0$ due to symmetry from the start, the solution actually follows more quickly.

$$x = a \cosh \frac{y}{a}$$

$$\frac{x}{a} = \cosh \frac{x}{a} \frac{y}{x}$$

Using, the dimensional quantities defined in the problem,

$$k = \frac{x_2}{a} \qquad \alpha = \frac{y_2}{x_2}$$

we have

$$k = \cosh k\alpha$$

Taking the derivative with respect to k,

$$1 = \alpha_0 \sinh k\alpha_0$$

Using the hyperbolic identity,

$$\cosh^2 A - \sinh^2 A = 1$$

a more manageable expression in terms of k and α becomes apparent,

$$k^2 - \frac{1}{\alpha_0^2} = 1$$

$$\alpha_0 = \frac{1}{\sqrt{k^2 - 1}}$$

Plug this into $k = \cosh k\alpha$

$$k = \cosh \frac{k}{\sqrt{k^2 - 1}}$$

Solving this numerically for k yields,

$$k \approx 1.81$$

Since

$$\alpha_0 = \frac{1}{\sqrt{k^2 - 1}} \quad \Rightarrow \quad \alpha_0 \approx .66$$

If $\alpha < \alpha_0$, two values of k are possible. If $\alpha > \alpha_0$, no real values of k exist, but if $\alpha = \alpha_0$ then only $k \approx 1.81$ will work. This graph is $arccosh(k)/k = \alpha$ and looks like a little hill. It can be graphed by typing acosh(x)/x on a free applet at http://www.tacoma.ctc.edu/home/jkim/gcalc.html.

2.12 The term *generalized mechanics* has come to designate a variety of classical mechanics in which the Lagrangian contains time derivatives of q_i higher than the first. Problems for which triple dot $x = f(x, \dot{x}, \ddot{x}, t)$ have been referred to as 'jerky' mechanics. Such equations of motion have interesting applications in chaos theory (cf. Chapter 11). By applying the mehtods of the calculus of variations, show that if there is a Lagrangian of the form $L(q_i, \dot{q}_i\ddot{q}_i, t)$, and Hamilton's principle holds with the zero variation of both q_i and $\dot{q}_i$ at the end points, then the corresponding Euler-Lagrange equations are

$$\frac{d^2}{dt^2}(\frac{\partial L}{\partial \ddot{q}_i}) - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, ..., n.$$

Apply this result to the Lagrangian

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$$

Do you recognize the equations of motion?

Answer:

If there is a Lagrangian of the form

$$L = L(q_i, \dot{q}_i, \ddot{q}_i, t)$$

and Hamilton's principle holds with the zero variation of both q_i and $\dot{q}_i$ at the end points, then we have:

$$I = \int_1^2 L(q_i, \dot{q}_i, \ddot{q}_i, t)dt$$

and

$$\frac{\partial I}{\partial \alpha}d\alpha = \int_1^2 \sum_i (\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha_i}d\alpha + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha_i}d\alpha + \frac{\partial L}{\partial \ddot{q}_i}\frac{\partial \ddot{q}_i}{\partial \alpha_i}d\alpha)dt$$

To make life easier, we're going to assume the Einstein summation convention, as well as drop the indexes entirely. In analogy with the differential quantity, Goldstein Equation (2.12), we have

$$\delta q = \frac{\partial q}{\partial \alpha}d\alpha$$

Applying this we have

$$\delta I = \int_1^2 (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha}d\alpha + \frac{\partial L}{\partial \ddot{q}}\frac{\partial \ddot{q}}{\partial \alpha}d\alpha)dt$$

The indexes are invisible and the two far terms are begging for some mathematical manipulation. Integration by parts on the middle term yields, in analogy to Goldstein page 44,

$$\int_1^2 \frac{\partial L}{\partial \dot{q}} \frac{\partial^2 q}{\partial \alpha \partial t} dt = \frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial \alpha}\bigg|_1^2 - \int_1^2 \frac{\partial q}{\partial \alpha} \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}}) dt$$

This first term on the right is zero because the condition exists that all the varied curves pass through the fixed end points and thus the partial derivative of q wrt to α at x_1 and x_2 vanish. Substituting back in, we have:

$$\delta I = \int_1^2 (\frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta q + \frac{\partial L}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} d\alpha) dt$$

Where we used the definition $\delta q = \frac{\partial q}{\partial \alpha} d\alpha$ again. Now the last term needs attention. This requires integration by parts twice. Here goes:

$$\int_1^2 \frac{\partial L}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} dt = \frac{\partial L}{\partial \ddot{q}} \frac{\partial^2 q}{\partial t \partial \alpha}\bigg|_1^2 - \int_1^2 \frac{\partial^2 q}{\partial t \partial \alpha} \frac{d}{dt} (\frac{\partial L}{\partial \ddot{q}}) dt$$

Where we used $\int v du = uv - \int v du$ as before. The first term vanishes once again, and we are still left with another integration by parts problem. Turn the crank again.

$$- \int_1^2 \frac{\partial^2 q}{\partial t \partial \alpha} \frac{d}{dt} (\frac{\partial L}{\partial \ddot{q}}) dt = \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \frac{\partial q}{\partial \alpha}\bigg|_1^2 - \int_1^2 - \frac{\partial q}{\partial \alpha} \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} dt$$

First term vanishes for the third time, and we have

$$\int_1^2 \frac{\partial L}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} dt = \int_1^2 \frac{\partial q}{\partial \alpha} \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} dt$$

Plugging back in finally, and using the definition of our δq, we get closer

$$\delta I = \int_1^2 (\frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta q + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} \delta q) dt$$

Gathering δq's, throwing our summation sign and index's back in, and applying Hamiliton's principle:

$$\delta I = \int_1^2 \sum_i (\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i}) \delta q_i dt = 0$$

We know that since q variables are independent, the variations δq_i are independent and we can apply the calculus of variations lemma, (Goldstein, Eq. 2.10) and see that $\delta I = 0$ requires that the coefficients of δq_i separately vanish, one by one:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} = 0 \qquad i = 1, 2, ...n.$$

Applying this result to the Lagrangian,

$$L = -\frac{1}{2}mq\ddot{q} - \frac{k}{2}q^2$$

yields

$$\frac{\partial L}{\partial q} = -\frac{1}{2}m\ddot{q} - kq$$

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = \frac{d}{dt}(\frac{d}{dt}(-\frac{1}{2}mq)) = \frac{d}{dt}(-\frac{1}{2}m\dot{q}) = -\frac{1}{2}m\ddot{q}$$

Adding them up:

$$-m\ddot{q} - kq = 0$$

This is interesting because this equation of motion is just Hooke's Law. This crazy looking Lagrangian yields the same equation for simple harmonic motion using the 'jerky' form of Lagrangian's equations. It's interesting to notice that if the familiar Lagrangian for a simple harmonic oscillator (SHO) plus an extra term is used, the original Lagrangian can be obtained.

$$L = L_{SHO} + \frac{d}{dt}(-\frac{mq\dot{q}}{2})$$

$$L = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2} + \frac{d}{dt}(-\frac{mq\dot{q}}{2})$$

$$L = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2} - \frac{mq\ddot{q}}{2} - \frac{m\dot{q}^2}{2}$$

$$L = -\frac{mq\ddot{q}}{2} - \frac{kq^2}{2}$$

This extra term, $\frac{d}{dt}(-\frac{mq\dot{q}}{2})$ probably represents constraint. The generalized force of constraint is the Lagrange multipliers term that is added to the original form of Lagrange's equations.