Homework 7: # 4.22, 5.15, 5.21, 5.23, Foucault pendulum

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4.22
A projectile is fired horizontally along Earth's surface. Show that to a first approximation the angular deviation from the direction of fire resulting from the Coriolis effect varies linearly with time at a rate

$$\omega \cos\theta$$

where ω is the angular frequency of Earth's rotation and θ is the co-latitude, the direction of deviation being to the right in the northern hemisphere.

Answer:

I'll call the angular deviation ψ. We are to find

$$\psi = \omega \cos\theta t$$

We know ω is directed north along the axis of rotation, that is, sticking out of the north pole of the earth. We know θ is the co-latitude, that is, the angle from the poles to the point located on the surface of the Earth. The latitude, λ is the angle from the equator to the point located on the surface of the Earth. $\lambda = \pi/2 - \theta$. Place ourselves in the coordinate system of whoever may be firing the projectile on the surface of the Earth. Call y' the horizontal direction pointing north (not toward the north pole or into the ground, but horizontally north), call x' the horizontal direction pointed east, and call z' the vertical direction pointed toward the sky.

With our coordinate system in hand, lets see where ω is. Parallel transport it to the surface and note that it is between y' and z'. If we are at the north pole, it is completely aligned with z', if we are at the equator, ω is aligned with y'. Note that the angle between z' and ω is the co-latitude, θ.(θ is zero at the north pole, when ω and z' are aligned). If we look at the components of ω, we can take a hint from Goldstein's Figure 4.13, that deflection of the horizontal trajectory in the northern hemisphere will depend on only the z' component of ω, labeled $\omega_{z'}$. Only ω_z is used for our approximation. It is clear that there is no component of ω in the x' direction. If we took into account the component in the y' direction we would have an effect causing the particle to move into the vertical direction, because the Coriolis effect is

$$F_c = -2m(\omega \times v)$$

and $\omega_y \times v$ would add a contribution in the z direction because our projectile is fired only along x' and y', that is, horizontally. So following Goldstein's figure, we shall only be concerned with ω_z. Our acceleration due to the Coriolis force is

$$a_c = -2(\omega \times v) = 2(v \times \omega)$$

The component of ω in the z' direction is $\omega_{z'} = \omega \cos\theta$. Thus the magnitude of the acceleration is

$$a_c = 2v\omega \cos\theta$$

The distance affected by this acceleration can be found through the equation of motion,

$$d = \frac{1}{2}a_c t^2 = v\omega \cos\theta t^2$$

And using a small angle of deviation, for ψ we can draw a triangle and note that the distance traveled by the projectile is just $x = vt$.

$$x\psi = d \quad \rightarrow \quad \psi = \frac{d}{x}$$

$$\psi = \frac{v\omega \cos\theta t^2}{vt} = \omega \cos\theta t$$

Therefore the angular deviation varies linearly on time with a rate of $\omega \cos\theta$. Note that there is no Coriolis effect at the equator when $\theta = \pi/2$, therefore no angular deviation.

5.15
Find the principal moments of inertia about the center of mass of a flat rigid body in the shape of a 45^o right triangle with uniform mass density. What are the principal axes?

Answer:

Using the moment of inertia formula for a lamina, which is a flat closed surface, (as explained on wolfram research) we can calculate the moment of inertia for the triangle, with it situated with long side on the x-axis, while the y-axis cuts through the middle. The off-diagonal elements of the inertia tensor vanish.

$$I_x = \int \sigma y^2 dxdy = 2\int_0^a \int_0^{a-x} \frac{M}{A} y^2 dydx = \frac{2M}{a^2}\int_0^a \frac{(a-x)^3}{3} dx$$

Solving the algebra,

$$I_x = \frac{2M}{3a^2}\int_0^a (-x^3 + 3ax^2 - 3a^2x + a^3)dx = \frac{2Ma^2}{3}[\frac{8}{4} - \frac{1}{4} - \frac{6}{4}] = \frac{Ma^2}{6}$$

For I_y

$$I_y = \int \sigma x^2 dxdy = 2\int_0^a \int_0^{a-y} \frac{M}{A} x^2 dxdy$$

This has the exact same form, so if you're clever, you won't do the integral over again.

$$I_y = \frac{Ma^2}{6}$$

For I_z

$$I_z = \int \sigma(x^2 + y^2)dxdy = I_x + I_y = (\frac{1}{6} + \frac{1}{6})Ma^2 = \frac{Ma^2}{3}$$

We can use the parallel axis theorem to find the principal moments of inertia about the center of mass. The center of mass is

$$y_{cm} = 2\frac{\sigma}{M}\int_0^a \int_0^{a-x} ydxdy = \frac{2}{a^2}\int_0^a \frac{(a-x)^2}{2} dx$$

$$y_{cm} = \frac{1}{a^2}\int_0^a (a^2 - 2xa + x^2)dx = a^2x - ax^2 + \frac{x^3}{3}\Big|_0^a \frac{1}{a^2} = \frac{a}{3}$$

From symmetry we can tell that the center of mass is $(0, \frac{a}{3}, 0)$. Using the parallel axis theorem, with $r_0 = a/3$

$$I_X = I_x - Mr_0^2$$
$$I_Y = I_y$$
$$I_Z = I_z - Mr_0^2$$

These are

$$I_X = (\frac{1}{6} - \frac{1}{9})Ma^2 = (\frac{3}{18} - \frac{2}{18})Ma^2 = \frac{Ma^2}{18}$$

$$I_Y = \frac{Ma^2}{6}$$

$$I_Z = (\frac{1}{3} - \frac{1}{9})Ma^2 = \frac{2}{9}Ma^2$$

5.21
A compound pendulum consists of a rigid body in the shape of a lamina suspended in the vertical plane at a point other than the center of gravity. Compute the period for small oscillations in terms of the radius of gyration about the center of gravity and the separation of the point of suspension from the center of gravity. Show that if the pendulum has the same period for two points of suspension at unequal distances from the center of gravity, then the sum of these distances is equal to the length of the equivalent simple pendulum.

Answer:

Looking for an equation of motion, we may equate the torque to the moment of inertia times the angular acceleration.

$$lF = I\ddot{\theta}$$

The force is $-Mg\sin\theta$, and the moment of inertia, using the parallel axis theorem is

$$I = Mr_g^2 + Ml^2$$

where r_g radius of gyration about the center of gravity, and l is the distance between the pivot point and center of gravity. The equation of motion becomes

$$-lMg\sin\theta = (Mr_g^2 + Ml^2)\ddot{\theta}$$

Using small oscillations, we can apply the small angle approximation $\sin\theta \approx \theta$

$$-lg\theta = (r_g^2 + l^2)\ddot{\theta}$$

$$\frac{lg}{r_g^2 + l^2}\theta + \ddot{\theta} = 0$$

This is with angular frequency and period

$$\omega = \sqrt{\frac{lg}{r_g^2 + l^2}} \quad \rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_g^2 + l^2}{lg}}$$

This is the same as the period for a physical pendulum

$$T = 2\pi\sqrt{\frac{I}{Mgl}} = 2\pi\sqrt{\frac{r_g^2 + l^2}{lg}}$$

If we have two points of suspension, l_1 and l_2, each having the same period, T. Then we get

$$2\pi\sqrt{\frac{r_g^2+l_1^2}{l_1 g}} = 2\pi\sqrt{\frac{r_g^2+l_2^2}{l_2 g}}$$

This is

$$\frac{r_g^2+l_1^2}{l_1} = \frac{r_g^2+l_2^2}{l_2}$$

And in a more favorable form, add l_1 to both sides, because we are looking for l_1+l_2 to be equivalent to a simple pendulum length,

$$\frac{r_g^2}{l_1}+l_1+l_1 = \frac{r_g^2}{l_2}+l_2+l_1$$

$$\frac{r_g^2}{l_1 l_2}(l_2-l_1)+2l_1 = l_2+l_1$$

This is only true if

$$r_g^2 = l_1 l_2$$

Thus our period becomes

$$T = 2\pi\sqrt{\frac{r_g^2+l_1^2}{l_1 g}} = 2\pi\sqrt{\frac{l_1 l_2+l_1^2}{l_1 g}} = 2\pi\sqrt{\frac{l_2+l_1}{g}} = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of a simple pendulum equivalent.

5.23
An automobile is started from rest with one of its doors initially at right angles. If the hinges of the door are toward the front of the car, the door will slam shut as the automobile picks up speed. Obtain a formula for the time needed for the door to close if the acceleration f is constant, the radius of gyration of the door about the axis of rotation is r_0 and the center of mass is at a distance a from the hinges. Show that if f is $0.3m/s^2$ and the door is a uniform rectangle is 1.2m wide, the time will be approximately 3.04 s.

Answer:

Begin by setting the torque equal to the product of the moment of inertia and angular acceleration.

$$I\ddot{\theta} = aF$$

The moment of inertia is $I = mr_0^2$. The force is $F = -mf\sin\theta$. So we get

$$mr_0^2\ddot{\theta} = -amf\sin\theta$$

Our equation of motion is

$$\ddot{\theta} = -\frac{af}{r_0^2}\sin\theta$$

This is rough. In our case we can not use the small angle approximation. The door starts at 90^o! How do we go about solving this then? Lets try integrating it once and see how far we can get. Here is a handy trick,

$$\ddot{\theta} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{d\theta}\dot{\theta}$$

Plug this into our equation of motion

$$\frac{d\dot{\theta}}{d\theta}\dot{\theta} = -\frac{af}{r_0^2}\sin\theta$$

This is separable, and may be integrated.

$$\frac{\dot{\theta}^2}{2} = \frac{af}{r_0^2}\cos\theta$$

$$\dot{\theta} = \sqrt{\frac{2af}{r_0^2}\cos\theta}$$

The time may be found by integrating over the time of travel it takes for the door to shut.

$$T = \int_0^{\frac{\pi}{2}} \frac{dt}{d\theta}d\theta = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\dot{\theta}} = \int_0^{\frac{\pi}{2}} \sqrt{\frac{r_0^2}{2af}}\frac{d\theta}{\sqrt{\cos\theta}}$$

Here is where the physics takes a backseat for a few, while the math runs the show. If we throw in a $-\cos 90^o$ we might notice that this integral is an elliptic integral of the first kind, denoted K.

$$T = \sqrt{\frac{r_0^2}{2af}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta}} = \sqrt{\frac{r_0^2}{2af}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta - \cos\frac{\pi}{2}}} = \sqrt{\frac{r_0^2}{2af}}\sqrt{2}K(\sin\frac{\pi}{4})$$

This can be seen from mathworld's treatment of elliptic integrals, at
http://mathworld.wolfram.com/EllipticIntegraloftheFirstKind.html.
Now we have

$$T = \sqrt{\frac{r_0^2}{af}}K(\frac{\sqrt{2}}{2})$$

$K(\frac{\sqrt{2}}{2})$ belongs to a group of functions called 'elliptic integral singular values', $K(k_r)$ A treatment of them and a table of their values that correspond to gamma functions are given here:
http://mathworld.wolfram.com/EllipticIntegralSingularValue.html.

The 'elliptic lambda function' determines the value of k_r. A table of lambda functions is here

http://mathworld.wolfram.com/EllipticLambdaFunction.html.

Our k_r value of $\frac{\sqrt{2}}{2}$ corresponds to k_1. From the singular value table,

$$K(k_1) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$

Our time is now

$$T = \sqrt{\frac{r_0^2}{af}} \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$

Fortunately, there are nice calculators that will compute gamma functions quickly. I used this one

http://www.efunda.com/math/gamma/findgamma.cfm.

I now have

$$\Gamma(\frac{1}{4}) = 3.63$$

Back to the physics. The moment of inertia of a uniform rectangle about the axis that bisects it is $\frac{M}{3}a^2$. Move the axis to the edge of the rectangle using the parallel axis theorem.

$$I = Mr_0^2 = Ma^2 + \frac{M}{3}a^2 = \frac{4}{3}Ma^2$$

we now have

$$r_0^2 = \frac{4}{3}a^2$$

With $a = .6m$, that is, half of the length of the car door, assuming its mass is uniform. And with $f = .3m/s^2$ we have

$$T = \sqrt{\frac{4a}{3f}} \frac{1}{4\sqrt{\pi}}(3.63)^2 = \sqrt{\frac{4(.6)}{3(.3)}} \frac{1}{4\sqrt{3.14}}(3.63)^2 = 3.035 \approx 3.04\ s$$

Foucault Pendulum
Find the period of rotation as a function of latitude.
Hint: neglect centrifugal force, neglect change in height, solve for $\xi = x + iy$

Answer:

The Foucault pendulum is a swinging weight supported by a long wire, so that the wire's upper support restrains the wire only in the vertical direction and the weight is set swinging with no lateral or circular motion. The plane of the pendulum gradually rotates, demonstrating the Earth's rotation. Solve for the period of rotation of this plane. The equation of motion for acceleration takes into account the vertical acceleration due to gravity, the acceleration from the tension and the Coriolis acceleration.

$$a_r = g + \frac{T}{m} - 2\omega \times v_r$$

In my system, I have x facing east, y facing north, and z facing to the sky. This yeilds

$$\omega_x = 0$$
$$\omega_y = \omega \sin\theta = \omega \cos\lambda$$
$$\omega_z = \omega \cos\theta = \omega \sin\lambda$$

The only velocity contributions come from the x and y components, for we can ignore the change in height. The Coriolis acceleration is quickly derived

$$a_c = \dot{y}\omega \sin\lambda \hat{x} - \dot{x}\omega \sin\lambda \hat{y} + \dot{x}\omega \cos\lambda \hat{z}$$

Looking for the period of rotation, we are concerned only with the x and y accelerations. Our overall acceleration equations become

$$\ddot{x} = -\frac{g}{l}x + 2\dot{y}\omega \sin\lambda$$
$$\ddot{y} = -\frac{g}{l}y - 2\dot{x}\omega \sin\lambda$$

The g/l terms were found using approximations for the tension components, that is, $T_x = -T\frac{x}{l} \rightarrow T_x/ml = g/l$ and the same for y.

Introducing $\xi = x + iy$ and adding the two equations after multiplying the second one by i

$$\ddot{\xi} + \frac{g}{l}\xi = -2\omega \sin\lambda(-\dot{y} + i\dot{x})$$
$$\ddot{\xi} + \frac{g}{l}\xi = -2i\omega \sin\lambda\dot{\xi}$$

$$\ddot{\xi} + \frac{g}{l}\xi + 2i\omega \sin\lambda\dot{\xi} = 0$$

This is the damped oscillation expression. It's solution is, using $\frac{g}{l} >> \omega \sin\lambda$, the over damped case

$$\xi = e^{-i\omega \sin\lambda t}(Ae^{i\sqrt{\frac{g}{l}}t} + Be^{-i\sqrt{\frac{g}{l}}t})$$

The equation for oscillation of a pendulum is

$$\ddot{q} + \frac{g}{l}q = 0$$

It has solution

$$q = Ae^{i\sqrt{\frac{g}{l}}t} + Be^{-i\sqrt{\frac{g}{l}}t}$$

We can simplify our expression then, using q

$$\xi = qe^{-i\omega \sin \lambda t}$$

Where the angular frequency of the plane's rotation is $\omega \cos \theta$, or $\omega \sin \lambda$ where λ is the latitude, and θ is the co-latitude. The period can be found using, $\omega = 2\pi/T$.

$$\frac{2\pi}{T_{earth}} \cos \theta = \frac{2\pi}{T_{Foucault}} \rightarrow T_{Foucault} = \frac{T_{Earth}}{\cos \theta}$$

This can be checked because we know the pendulum rotates completely in 1 day at the North pole where $\theta = 0$ and has no rotation at the equator where $\theta = 90^0$. Chapel Hill has a latitude of 36^o, a Foucault pendulum takes

$$T_{Foucault} = \frac{24\ hours}{\sin 36^o} \approx 41\ hours$$

to make a full revolution.