Homework 10: # 9.2, 9.6, 9.16, 9.31

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9.2
Show that the transformation for a system of one degree of freedom,

$$Q = q\cos\alpha - p\sin\alpha$$

$$P = q\sin\alpha + p\cos\alpha$$

satisfies the symplectic condition for any value of the parameter α. Find a generating function for the transformation. What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$? Does your generating function work for both of these cases?

Answer:

The symplectic condition is met if

$$MJ\tilde{M} = J$$

We can find M from

$$\dot{\zeta} = M\dot{\eta}$$

which is

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

We know J to be

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Solving $MJ\tilde{M}$ we get

$$M(J\tilde{M}) = M \begin{pmatrix} -\sin\alpha & \cos\alpha \\ -\cos\alpha & -\sin\alpha \end{pmatrix}$$

$$MJ\tilde{M} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -\sin\alpha & \cos\alpha \\ -\cos\alpha & -\sin\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Therefore

$$M J \tilde{M} = J$$

and the symplectic condition is met for this transformation. To find the generating function, I will first attempt an F_1 type and proceed to solve, and check at the end if there are problems with it. Rearranging to solve for $p(Q,q)$ we have

$$p = -\frac{Q}{\sin\alpha} + \frac{q\cos\alpha}{\sin\alpha}$$

The related equation for F_1 is

$$p = \frac{\partial F_1}{\partial q}$$

Integrating for F_1 yields

$$F_1 = -\frac{Qq}{\sin\alpha} + \frac{q^2\cos\alpha}{2\sin\alpha} + g(Q)$$

Solve the other one, that is $P(Q,q)$, it along with its relevant equation is

$$P = q\sin\alpha - \frac{Q\cos\alpha}{\sin\alpha} + \frac{q\cos^2\alpha}{\sin\alpha}$$

$$P = -\frac{\partial F_1}{\partial Q}$$

Integrating

$$-F_1 = qQ\sin\alpha - \frac{Q^2}{2}\cot\alpha + qQ(\frac{1}{\sin\alpha} - \sin\alpha) + h(q)$$

$$-F_1 = -\frac{Q^2}{2}\cot\alpha + \frac{qQ}{\sin\alpha} + h(q)$$

$$F_1 = \frac{Q^2}{2}\cot\alpha - \frac{qQ}{\sin\alpha} + h(q)$$

Using both F_1's we find

$$F_1 = -\frac{Qq}{\sin\alpha} + \frac{1}{2}(q^2 + Q^2)\cot\alpha$$

This has a problem. It blows up, sky high, when $\alpha = n\pi$. But otherwise its ok, lets put the condition, $\alpha \neq n\pi$. If we solve for F_2 we may be able to find out what the generating function is, and have it work for the holes, $\alpha = n\pi$. $F_2(q,P,t)$'s relevant equations are

$$p = \frac{\partial F_2}{\partial q}$$

$$p = \frac{P}{\cos\alpha} - \frac{q\sin\alpha}{\cos\alpha}$$

$$F_2 = \frac{Pq}{\cos\alpha} - \frac{q^2}{2}\tan\alpha + f(P)$$

and

$$Q = \frac{\partial F_2}{\partial P}$$

$$Q = q\cos\alpha - (P - q\sin\alpha)\tan\alpha$$

$$F_2 = qP\cos\alpha - \frac{P^2}{2}\tan\alpha + qP\frac{\sin^2\alpha}{\cos\alpha} + g(q)$$

$$F_2 = qP(\cos\alpha + \frac{1}{\cos\alpha} - \cos\alpha) - \frac{P^2}{2}\tan\alpha + g(q)$$

$$F_2 = \frac{qP}{\cos\alpha} - \frac{P^2}{2}\tan\alpha + g(q)$$

So therefore

$$F_2 = -\frac{1}{2}(q^2 + P^2)\tan\alpha + \frac{qP}{\cos\alpha}$$

This works for $\alpha = n\pi$ but blows sky high for $\alpha = (n+\frac{1}{2})\pi$. So I'll put a condition on F_2 that $\alpha \neq (n+\frac{1}{2})\pi$. The physical significance of this transformation for $\alpha = 0$ is easy to see cause we get

$$Q = q\cos 0 - p\sin 0 = q$$

$$P = q\sin 0 - p\cos 0 = p$$

This is just the identity transformation, or no rotation. For $\alpha = \pi/2$ we get

$$Q = q\cos\frac{\pi}{2} - p\sin\frac{\pi}{2} = -p$$

$$P = q\sin\frac{\pi}{2} - p\cos\frac{\pi}{2} = q$$

Where the p's and q's have been exchanged.

9.6 The transformation equations between two sets of coordinates are

$$Q = log(1 + q^{1/2}\cos p)$$
$$P = 2(1 + q^{1/2}\cos p)q^{1/2}\sin p$$

- Show directly from these transformation equations that Q, P are canonical variables if q and p are.
- Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p$$

Answer:

Q and P are considered canonical variables if these transformation equations satisfy the symplectic condition.

$$MJ\tilde{M} = J$$

Finding M:

$$\dot{\zeta} = M\dot{\eta}$$

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = M \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \qquad M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$$

$$\frac{\partial Q}{\partial q} = \frac{q^{-1/2}\cos p}{2(1 + q^{1/2}\cos p)}$$

$$\frac{\partial Q}{\partial p} = \frac{-q^{1/2}\sin p}{1 + q^{1/2}\cos p}$$

$$\frac{\partial P}{\partial q} = q^{-1/2}\sin p + 2\cos p \sin p$$

$$\frac{\partial P}{\partial p} = 2q^{1/2}\cos p + 2q\cos^2 p - 2q\sin^2 p$$

Remembering

$$\cos^2 A - \sin^2 A = \cos 2A \quad and \quad 2\sin A\cos A = \sin 2A$$
$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

we can proceed with ease.

$$JM = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & \frac{-q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix}$$

$$JM = \begin{pmatrix} q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \\ -\frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix}$$

Now

$$\tilde{M}JM = \begin{pmatrix} \frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & q^{-1/2}\sin p + \sin 2p \\ \frac{-q^{1/2}\sin p}{1+q^{1/2}\cos p} & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix} \begin{pmatrix} q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \\ -\frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)} & \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix}$$

You may see that the diagonal terms disappear, and we are left with some algebra for the off-diagonal terms.

$$\tilde{M}JM = \begin{pmatrix} 0 & messy \\ ugly & 0 \end{pmatrix}$$

Lets solve for *ugly*.

$$ugly = \frac{-q^{1/2}\sin p}{1+q^{1/2}\cos p}(q^{-1/2}\sin p + \sin 2p) - \frac{q^{-1/2}\cos p}{2(1+q^{1/2}\cos p)}(2q^{1/2}\cos p + 2q\cos 2p)$$

$$ugly = \frac{-\sin^2 p - q^{1/2}\sin p \sin 2p - \cos^2 p - q^{1/2}\cos p\cos 2p}{1+q^{1/2}\cos p}$$

$$ugly = \frac{-(1 + q^{1/2}(\cos 2p\cos p + \sin 2p \sin p))}{1+q^{1/2}\cos p}$$

$$ugly = \frac{-(1+q^{1/2}\cos p)}{1+q^{1/2}\cos p} = -1$$

Not so ugly anymore, eh? Suddenly ugly became pretty. The same works for *messy* except it becomes positive 1 because it has no negative terms out front. So finally we get

$$\tilde{M}JM = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

which is the symplectic condition, which proves Q and P are canonical variables. To show that

$$F_3 = -(e^Q - 1)^2 \tan p$$

generates this transformation we may take the relevant equations for F_3, solve them, and then solve for our transformation equations.

$$q = -\frac{\partial F_3}{\partial p} = -[-(e^Q - 1)^2 \sec^2 p]$$

$$P = -\frac{\partial F_3}{\partial Q} = -[-2(e^Q - 1)\tan p]e^Q$$

Solving for Q

$$q = (e^Q - 1)^2 \sec^2 p$$

$$1 + \frac{\sqrt{q}}{\sqrt{\sec^2 p}} = e^Q$$

$$Q = \ln(1 + q^{1/2}\cos p)$$

This is one of our transformation equations, now lets plug this into the expression for P and put P in terms of q and p to get the other one.

$$P = 2(1 + q^{1/2}\cos p - 1)\tan p(1 + q^{1/2}\cos p)$$

$$P = 2q^{1/2}\sin p(1 + q^{1/2}\cos p)$$

Thus F_3 is the generating function of our transformation equations.

9.16
For a symmetric rigid body, obtain formulas for evaluating the Poisson brackets

$$[\dot{\phi}, f(\theta, \phi, \psi)] \qquad [\dot{\psi}, f(\theta, \phi, \psi)]$$

where θ, ϕ, and ψ are the Euler angles, and f is any arbitrary function of the Euler angles.

Answer:

Poisson brackets are defined by

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

From Goldstein's section on Euler angles, we learned

$$\dot{\phi} = \frac{I_1 b - I_1 a\cos\theta}{I_1 \sin^2\theta} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

So calculating

$$[\dot{\phi}, f] = [\frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}, f]$$

Note that $f = f(\theta, \phi, \psi)$ and not of momenta. So our definition becomes

$$[\dot{\phi}, f] = -\frac{\partial \dot{\phi}}{\partial p_i} \frac{\partial f}{\partial q_i}$$

Taking only two derivatives because $\dot{\phi}$ doesn't depend on p_θ. We get

$$[\dot{\phi}, f] = (-\frac{1}{I_1 \sin^2 \theta} \frac{\partial f}{\partial \phi}) + (\frac{\cos \theta}{I_1 \sin^2 \theta} \frac{\partial f}{\partial \psi})$$

$$[\dot{\phi}, f] = \frac{1}{I_1 \sin^2 \theta}(-\frac{\partial f}{\partial \psi} + \frac{\partial f}{\partial \psi} \cos \theta)$$

For the next relation,

$$[\dot{\psi}, f] = -\frac{\partial \dot{\psi}}{\partial p_i} \frac{\partial f}{\partial q_i}$$

and

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi \cos \theta}{I_1 \sin^2 \theta} \cos \theta$$

This yields

$$[\dot{\psi}, f] = -(\frac{1}{I_3} + \frac{\cos^2 \theta}{I_1 \sin^2 \theta}) \frac{\partial f}{\partial \psi} + -(-\frac{\cos \theta}{I_1 \sin^2 \theta} \frac{\partial f}{\partial \psi})$$

$$[\dot{\psi}, f] = -(\frac{I_1 \sin^2 \theta}{I_3 I_1 \sin^2 \theta} + \frac{I_3 \cos^2 \theta}{I_3 I_1 \sin^2 \theta}) \frac{\partial f}{\partial \psi} + \frac{I_3 \cos \theta}{I_3 I_1 \sin^2 \theta} \frac{\partial f}{\partial \phi}$$

$$[\dot{\psi}, f] = \frac{1}{I_3 I_1 \sin^2 \theta}(I_3 \cos \theta \frac{\partial f}{\partial \phi} - (I_3 \cos^2 \theta + I_1 \sin^2 \theta) \frac{\partial f}{\partial \psi})$$

Both together, in final form

$$[\dot{\phi}, f] = \frac{1}{I_1 \sin^2 \theta}(-\frac{\partial f}{\partial \psi} + \frac{\partial f}{\partial \psi} \cos \theta)$$

$$[\dot{\psi}, f] = \frac{1}{I_3 I_1 \sin^2 \theta}(I_3 \cos \theta \frac{\partial f}{\partial \phi} - (I_3 \cos^2 \theta + I_1 \sin^2 \theta) \frac{\partial f}{\partial \psi})$$

9.31
Show by the use of Poisson brackets that for one-dimensional harmonic oscillator there is a constant of the motion u defined as

$$u(q, p, t) = \ln(p + im\omega q) - i\omega t, \omega = \sqrt{\frac{k}{m}}$$

What is the physical significance of this constant of motion?

Answer:

We have

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

which we must prove equals zero if u is to be a constant of the motion. The Hamiltonian is

$$H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2}$$

So we have

$$\frac{du}{dt} = \frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = \frac{im\omega}{p + im\omega q}(\frac{p}{m}) - \frac{1}{p + im\omega q}(kq) - i\omega$$

$$\frac{du}{dt} = \frac{i\omega p - kq}{p + im\omega q} - i\omega = \frac{i\omega p - m\omega^2 q}{p + im\omega q} - i\omega$$

$$\frac{du}{dt} = i\omega\frac{p + i\omega mq}{p + im\omega q} - i\omega = i\omega - i\omega$$

$$\frac{du}{dt} = 0$$

Its physical significance relates to phase.

Show Jacobi's Identity holds. Show

$$[f, gh] = g[f, h] + [f, g]h$$

where the brackets are Poisson.

Answer:

Goldstein verifies Jacobi's identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

using an efficient notation. I will follow his lead. If we say

$$u_i \equiv \frac{\partial u}{\partial \eta_i} \qquad v_{ij} \equiv \frac{\partial v}{\partial \eta_i \partial \eta_j}$$

Then a simple way of expressing the Poisson bracket becomes apparent

$$[u, v] = u_i J_{ij} v_j$$

This notation becomes valuable when expressing the the double Poisson bracket. Here we have

$$[u,[v,w]] = u_i J_{ij}[v,w]_j = u_i J_{ij}(v_k J_{kl} w_l)_j$$

Taking the partial with respect to η_j we use the product rule, remembering J_{kl} are just constants,

$$[u,[v,w]] = u_i J_{ij}(v_{kj} J_{kl} w_l + v_k J_{kl} w_{lj})$$

doing this for the other two double Poisson brackets, we get 4 more terms, for a total of 6. Looking at one double partial term, w we see there are only two terms that show up

$$J_{ij} J_{kl} u_i v_k w_{lj} \quad and \quad J_{ji} J_{kl} u_i v_k w_{jl}$$

The first from $[u,[v,w]]$ and the second from $[v,[w,u]]$. Add them up, realizing order of partial is immaterial, and J is antisymmetric:

$$(J_{ij} + J_{ji}) J_{kl} u_i v_k w_{lj} = 0$$

All the other terms are made of second partials of u or v and disappear in the same manner. Therefore

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

Its ok to do the second property the long way:

$$[f,gh] = \frac{\partial f}{\partial q_i}\frac{\partial (gh)}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial (gh)}{\partial q_i}$$

$$[f,gh] = \frac{\partial f}{\partial q_i}(\frac{\partial g}{\partial p_i}h + g\frac{\partial h}{\partial p_i}) - \frac{\partial f}{\partial p_i}(g\frac{\partial h}{\partial q_i} + \frac{\partial g}{\partial q_i}h)$$

Grouping terms

$$[f,gh] = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}h - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}h + g\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - g\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}$$

$$[f,gh] = [f,g]h + g[f,h]$$