# Homework 5: # 3.31, 3.32, 3.7a

Michael Good

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3.7a Show that the angle of recoil of the target particle relative to the incident direction of the scattered particle is simply $\Phi = \frac{1}{2}(\pi - \Theta)$.

Answer:

It helps to draw a figure for this problem. I don't yet know how to do this in LaTeX, but I do know that in the center of mass frame both the particles momentum are equal.

$$m_1 v_1' = m_2 v_2'$$

Where the prime indicates the CM frame. If you take equation (3.2) Goldstein, then its easy to understand the equation after (3.110) for the relationship of the relative speed $v$ after the collision to the speed in the CM system.

$$v_1' = \frac{\mu}{m_1} v = \frac{m_2}{m_1 + m_2} v$$

Here, $v$ is the relative speed *after* the collision, but as Goldstein mentions because elastic collisions conserve kinetic energy, (I'm assuming this collision is elastic even though it wasn't explicitly stated), we have $v = v_0$, that is the relative speed after collision is equal to the initial velocity of the first particle in the laboratory frame ( the target particle being stationary).

$$v_1' = \frac{m_2}{m_1 + m_2} v_0$$

This equation works the same way for $v_2'$

$$v_2' = \frac{m_1}{m_1 + m_2} v_0$$

From conservation of momentum, we know that the total momentum in the CM frame is equal to the incident(and thus total) momentum in the laboratory frame.

$$(m_1 + m_2) v_{cm} = m_1 v_0$$

We see

$$v_{cm} = \frac{m_1}{m_1 + m_2} v_0$$

This is the same as $v_2'$

$$v_2' = v_{cm}$$

If we draw both frames in the same diagram, we can see an isosceles triangle where the two equal sides are $v_2'$ and $v_{cm}$.

$$\Phi + \Phi + \Theta = \pi$$

$$\Phi = \frac{1}{2}(\pi - \Theta)$$

3.31 Examine the scattering produced by a repulsive central force $f + kr^{-3}$. Show that the differential cross section is given by

$$\sigma(\Theta)d\Theta = \frac{k}{2E} \frac{(1-x)dx}{x^2(2--x)^2 \sin \pi x}$$

where $x$ is the ratio of $\Theta/\pi$ and $E$ is the energy.

Answer:

The differential cross section is given by Goldstein (3.93):

$$\sigma(\Theta) = \frac{s}{\sin \Theta} \left| \frac{ds}{d\Theta} \right|$$

We must solve for $s$, and $ds/d\Theta$. Lets solve for $\Theta(s)$ first, take its derivative with respect to $s$, and invert it to find $ds/d\Theta$. We can solve for $\Theta(s)$ by using Goldstein (3.96):

$$\Theta(s) = \pi - 2 \int_{r_m}^{\infty} \frac{sdr}{r\sqrt{r^2(1 - \frac{V(r)}{E}) - s^2}}$$

What is $V(r)$ for our central force of $f = k/r^3$? It's found from $-dV/dr = f$.

$$V(r) = \frac{k}{2r^2}$$

Plug this in to $\Theta$ and we have

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{sdr}{r\sqrt{r^2 - (s^2 + \frac{k}{2E})}}$$

Before taking this integral, I'd like to put it in a better form. If we look at the energy of the incoming particle,

$$E = \frac{1}{2}mr_m^2\dot{\theta}^2 + \frac{k}{2r_m^2} = \frac{s^2E}{r_m^2} + \frac{k}{2r_m^2}$$

where from Goldstein page 113,

$$\dot{\theta}^2 = \frac{2s^2E}{mr_m^4}$$

We can solve for $s^2 + \frac{k}{2E}$, the term in $\Theta$,

$$r_m^2 = s^2 + \frac{k}{2E}$$

Now we are in a better position to integrate,

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{sdr}{r\sqrt{r^2 - r_m^2}} = \pi - 2s\,[\frac{1}{r_m}\cos^{-1}\frac{r_m}{r}\Big|_{r_m}^{\infty}] = \pi - 2s\frac{1}{r_m}(\frac{\pi}{2}) = \pi(1 - \frac{s}{\sqrt{s^2 + \frac{k}{2E}}})$$

Goldstein gave us $x = \Theta/\pi$, so now we have an expression for $x$ in terms of $s$, lets solve for $s$

$$x = \frac{\Theta}{\pi} = 1 - \frac{s}{\sqrt{s^2 + \frac{k}{2E}}}$$

$$s^2 = (s^2 + \frac{k}{2E})(1-x)^2 \quad \rightarrow \quad s^2 = \frac{\frac{k}{2E}(1-x)^2}{1-(1-x)^2}$$

$$s = \sqrt{\frac{k}{2E}}\frac{(1-x)}{\sqrt{x(2-x)}}$$

Now that we have $s$ we need only $ds/d\Theta$ to find the cross section. Solving $d\Theta/ds$ and then taking the inverse,

$$\frac{d\Theta}{ds} = \pi s(-\frac{1}{2}(s^2 + \frac{k}{2E})^{-\frac{3}{2}})2s + \frac{\pi}{\sqrt{s^2 + \frac{k}{2E}}}$$

$$\frac{d\Theta}{ds} = \frac{-\pi s^2 + \pi(s^2 + \frac{k}{2E})}{(s^2 + \frac{k}{2E})^{\frac{3}{2}}} = \frac{\frac{\pi k}{2E}}{(s^2 + \frac{k}{2E})^{\frac{3}{2}}}$$

So

$$\frac{ds}{d\Theta} = \frac{2E(s^2 + \frac{k}{2E})^{\frac{3}{2}}}{\pi k}$$

Putting everything in terms of $x$,

$$s^2 + \frac{k}{2E} = \frac{k}{2E}\frac{(1-x)^2}{x(2-x)} + \frac{k}{2E} = \frac{k}{2E}\frac{1}{x(2-x)}$$

So now,

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| = \frac{\sqrt{\frac{k}{2E}}\frac{(1-x)}{\sqrt{x(2-x)}}}{\sin\pi x}\frac{2E(s^2+\frac{k}{2E})^{\frac{3}{2}}}{\pi k} = \frac{\sqrt{\frac{k}{2E}}\frac{(1-x)}{\sqrt{x(2-x)}}}{\sin\pi x}\frac{2E(\frac{k}{2E}\frac{1}{x(2-x)})^{\frac{3}{2}}}{\pi k} =$$

And this most beautiful expression becomes..

$$\sigma(\Theta) = \frac{1}{\sin\pi x}\frac{1}{\pi}(\frac{k}{2E})^{\frac{1}{2}}(\frac{2E}{k})(\frac{k}{2E})^{\frac{3}{2}}\frac{1-x}{\sqrt{x(2-x)}}\frac{1}{(x(2-x))^{\frac{3}{2}}}$$

After a bit more algebra...

$$\sigma(\Theta) = \frac{k}{2E}\frac{1}{\pi}\frac{1}{\sin\pi x}\frac{1-x}{(x(2-x))^2}$$

And since we know $d\Theta = \pi dx$,

$$\sigma(\Theta)d\Theta = \frac{k}{2E}\frac{(1-x)dx}{x^2(2-x)^2\sin\pi x}$$

3.32 A central force potential frequently encountered in nuclear physics is the rectangular well, defined by the potential

$$V = 0 \qquad r > a$$

$$V = -V_0 \qquad r \leq a$$

Show that the scattering produced by such a potential in classical mechanics is identical with the refraction of light rays by a sphere of radius $a$ and relative index of refraction

$$n = \sqrt{\frac{E+V_0}{E}}$$

This equivalence demonstrates why it was possible to explain refraction phenomena both by Huygen's waves and by Newton's mechanical corpuscles. Show also that the differential cross section is

$$\sigma(\Theta) = \frac{n^2a^2}{4\cos\frac{\Theta}{2}}\frac{(n\cos\frac{\Theta}{2}-1)(n-\cos\frac{\Theta}{2})}{(1+n^2-2n\cos\frac{\Theta}{2})^2}$$

What is the total cross section?

Answer:

Ignoring the first part of the problem, and just solving for the differential cross section,

$$\sigma(\Theta) = \frac{sds}{\sin\Theta d\Theta}$$

If the scattering is the same as light refracted from a sphere, then putting our total angle scattered, $\Theta$, in terms of the angle of incidence and transmission,

$$\Theta = 2(\theta_1 - \theta_2)$$

This is because the light is refracted from its horizontal direction twice, after hitting the sphere and leaving the sphere. Where $\theta_1 - \theta_2$ is the angle south of east for one refraction.

We know $\sin\theta_1 = s/a$ and using Snell's law, we know

$$n = \frac{\sin\theta_1}{\sin\theta_2} \rightarrow \sin\theta_2 = \frac{s}{na}$$

Expressing $\Theta$ in terms of just $s$ and $a$ we have

$$\Theta = 2(\arcsin\frac{s}{a} - \arcsin\frac{s}{na})$$

Now the plan is, to solve for $s^2$ and then $ds^2/d\Theta$ and solve for the cross section via

$$\sigma = \frac{sds}{\sin\Theta d\Theta} = \frac{1}{2\sin\Theta}\frac{ds^2}{d\Theta} = \frac{1}{4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{ds^2}{d\Theta}$$

Here goes. Solve for $\sin\frac{\Theta}{2}$ and $\cos\frac{\Theta}{2}$ in terms of $s$

$$\sin\frac{\Theta}{2} = \sin(\arcsin\frac{s}{a} - \arcsin\frac{s}{na}) = \sin\arcsin\frac{s}{a}\cos\arcsin\frac{s}{na} - \cos\arcsin\frac{s}{a}\sin\arcsin\frac{s}{na}$$

This is

$$= \frac{s}{a}\cos(\arccos\sqrt{1-\frac{s^2}{n^2a^2}}) - \cos(\arccos(\sqrt{1-\frac{s^2}{a^2}})\frac{s}{na}$$

Using $\arcsin x = \arccos\sqrt{1-x^2}$ and $\sin(a-b) = \sin a\cos b - \cos a\sin b$. Now we have

$$\sin\frac{\Theta}{2} = \frac{s}{na^2}(\sqrt{n^2a^2-s^2}) - \sqrt{a^2-s^2})$$

Doing the same thing for $\cos\frac{\Theta}{2}$ yields

$$\cos\frac{\Theta}{2} = \frac{1}{na^2}(\sqrt{a^2-s^2}\sqrt{n^2a^2-s^2}+s^2)$$

Using $\cos(a-b) = \cos a\cos b + \sin a \sin b$. Still solving for $s^2$ in terms of cos and sin's we proceed

$$\sin^2\frac{\Theta}{2} = \frac{s^2}{n^2a^4}(n^2a^2 - s^2 - 2\sqrt{n^2a^2-s^2}\sqrt{a^2-s^2} + a^2 - s^2)$$

This is

$$\sin^2\frac{s^2}{n^2a^2}(n^2+1) - \frac{2s^4}{n^2a^4} - \frac{2s^2}{n^2a^4}\sqrt{n^2a^2-s^2}\sqrt{a^2-s^2}$$

Note that

$$\sqrt{n^2a^2-s^2}\sqrt{a^2-s^2} = na^2\cos\frac{\Theta}{2} - s^2$$

So we have

$$\sin^2\frac{\Theta}{2} = \frac{s^2}{n^2a^2}(n^2+1-\frac{2s^2}{a^2} - 2n\cos\frac{\Theta}{2} + \frac{2s^2}{a^2}) = \frac{s^2}{n^2a^2}(1+n^2-2n\cos\frac{\Theta}{2})$$

Solving for $s^2$

$$s^2 = \frac{n^2a^2\sin^2\frac{\Theta}{2}}{1+n^2-2n\cos\frac{\Theta}{2}}$$

Glad that that mess is over with, we can now do some calculus. I'm going to let $q^2$ equal the denominator squared. Also to save space, lets say $\frac{\Theta}{2} = Q$. I like using the letter $q$.

$$\frac{ds^2}{d\Theta} = \frac{a^2\sin Qn^2}{q^2}[\cos Q(1-2n\cos Q + n^2) - n\sin^2 Q]$$

$$\frac{ds^2}{d\Theta} = \frac{n^2a^2}{q^2}\sin Q[\cos Q - 2n\cos^2 Q + n^2\cos Q - n(1-\cos^2 Q)]$$

Expand and collect

$$\frac{ds^2}{d\Theta} = \frac{n^2a^2}{q^2}\sin Q[-n\cos^2 Q + \cos Q + n^2\cos Q - n]$$

Group it up

$$\frac{ds^2}{d\Theta} = \frac{n^2a^2}{q^2}\sin Q(n\cos Q - 1)(n - \cos Q)$$

Plug back in for $Q$ and $q^2$:

$$\frac{ds^2}{d\Theta} = \frac{n^2a^2\sin\frac{\Theta}{2}(n\cos\frac{\Theta}{2}-1)(n-\cos\frac{\Theta}{2})}{(1-2n\cos\frac{\Theta}{2}+n^2)^2}$$

Using our plan from above,

$$\sigma = \frac{1}{4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{ds^2}{d\Theta} = \frac{1}{4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{n^2a^2\sin\frac{\Theta}{2}(n\cos\frac{\Theta}{2}-1)(n-\cos\frac{\Theta}{2})}{(1-2n\cos\frac{\Theta}{2}+n^2)^2}$$

We obtain

$$\sigma(\Theta) = \frac{1}{4\cos\frac{\Theta}{2}}\frac{n^2a^2(n\cos\frac{\Theta}{2}-1)(n-\cos\frac{\Theta}{2})}{(1-2n\cos\frac{\Theta}{2}+n^2)^2}$$

The total cross section involves an algebraic intensive integral. The total cross section is given by

$$\sigma_T = 2\pi\int_0^{\Theta_{max}}\sigma(\Theta)\sin\Theta d\Theta$$

To find $\Theta_{max}$ we look for when the cross section becomes zero. When $(n\cos\frac{\Theta}{2}-1)$ is zero, we'll have $\Theta_{max}$. If $s > a$, its as if the incoming particle misses the 'sphere'. At $s = a$ we have maximum $\Theta$. So using $\Theta_{max} = 2\arccos\frac{1}{n}$, we will find it easier to plug in $x = \cos\frac{\Theta}{2}$ as a substitution, to simplify our integral.

$$\sigma_T = \pi\int_{\frac{1}{n}}^{1} a^2n^2\frac{(nx-1)(n-x)}{(1-2nx+n^2)^2}2dx$$

where

$$dx = -\frac{1}{2}\sin\frac{\Theta}{2}d\Theta \qquad \cos\frac{\Theta_{max}}{2} = \frac{1}{n}$$

The half angle formula, $\sin\Theta = 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}$ was used on the $\sin\Theta$, the negative sign switched the direction of integration, and the factor of 2 had to be thrown in to make the $dx$ substitution.

This integral is still hard to manage, so make another substitution, this time, let $q$ equal the term in the denominator.

$$q = 1-2nx+n^2 \qquad \rightarrow \qquad dq = -2ndx$$

The algebra must be done carefully here. Making a partial substitution to see where to go:

$$q_{min} = 1-2+n^2 = n^2-1 \qquad q_{max} = n^2-2n+1 = (n-1)^2$$

$$\sigma_T = \int_{n^2-1}^{(n-1)^2}\frac{2\pi a^2n^2(nx-1)(n-x)}{q^2}\frac{dq}{-2n} = \pi a^2\int_{n^2-1}^{(n-1)^2}\frac{-n(nx-1)(n-x)}{q^2}dq$$

Expanding $q^2$ to see what it gives so we can put the numerator in the above integral in terms of $q^2$ we see

$$q^2 = n^4 + 1 + 2n^2 - 4n^3x - 4nx + 4n^2x^2$$

Expanding the numerator

$$-n(nx-1)(n-x) = -n^3x - nx + n^2x^2 + n^2$$

If we take $q^2$ and subtract a $n^4$, subtract a 1, add a $2n^2$ and divide the whole thing by 4 we'll get the above numerator. That is:

$$\frac{q^2 - n^4 + 2n^2 - 1}{4} = \frac{q^2 - (n^2-1)^2}{4} = -n(nx-1)(n-x)$$

Now, our integral is

$$\sigma_T = \pi a^2 \int_{n^2-1}^{(n-1)^2} \frac{q^2 - (n^2-1)^2}{4q^2} dq$$

This is finally an integral that can be done by hand

$$\sigma_T = \frac{\pi a^2}{4} \int 1 - \frac{(n^2-1)^2}{q^2} dq = \frac{\pi a^2}{4} \, (z + \frac{(n^2-1)^2}{z} \Big|_{n^2-1}^{(n-1)^2})$$

After working out the few steps of algebra,

$$\frac{\pi a^2}{4} \frac{4n^2 - 8n + 4}{n^2 - 2n + 1} = \pi a^2$$

The total cross section is

$$\sigma_T = \pi a^2$$