Homework 8: # 5.4, 5.6, 5.7, 5.26

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5.4
Derive Euler's equations of motion, Eq. (5.39'), from the Lagrange equation of motion, in the form of Eq. (1.53), for the generalized coordinate ψ.

Answer:

Euler's equations of motion for a rigid body are:

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

The Lagrangian equation of motion is in the form

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j$$

The kinetic energy for rotational motion is

$$T = \sum_i^3 \frac{1}{2} I_i \omega_i^2$$

The components of the angular velocity in terms of Euler angles for the body set of axes are

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

Solving for the equation of motion using the generalized coordinate ψ:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\psi}}) - \frac{\partial T}{\partial \psi} = N_\psi$$

$$\sum_i^3 I_i\omega_i \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt}\sum_i^3 I_i\omega_i \frac{\partial \omega_i}{\partial \dot{\psi}} = N_\psi$$

Now is a good time to pause and calculate the partials of the angular velocities,

$$\frac{\partial\omega_1}{\partial\psi} = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$

$$\frac{\partial\omega_2}{\partial\psi} = -\dot{\theta}\cos\psi - \dot{\phi}\sin\theta\sin\psi$$

$$\frac{\partial\omega_3}{\partial\psi} = 0$$

and

$$\frac{\partial\omega_1}{\partial\dot{\psi}} = \frac{\partial\omega_2}{\partial\dot{\psi}} = 0$$

$$\frac{\partial\omega_3}{\partial\dot{\psi}} = 1$$

Now we have all the pieces of the puzzle, explicitly

$$\sum_i^3 I_i\omega_i\frac{\partial\omega_i}{\partial\psi} - \frac{d}{dt}\sum_i^3 I_i\omega_i\frac{\partial\omega_i}{\partial\dot{\psi}} = N_\psi$$

$$I_1\omega_1(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi) + I_2\omega_2(-\dot{\theta}\cos\psi - \dot{\phi}\sin\theta\sin\psi) - \frac{d}{dt}I_3\omega_3 = N_\psi$$

This is, pulling out the negative sign on the second term,

$$I_1\omega_1(\omega_2) - I_2\omega_2(\omega_1) - I_3\dot{\omega}_3 = N_\psi$$

$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_\psi$$

And through cyclic permutations

$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$

we have the rest of Euler's equations of motion for a rigid body.

5.6

- Show that the angular momentum of the torque-free symmetrical top rotates in the body coordinates about the symmetry axis with an angular frequency ω. Show also that the symmetry axis rotates in space about the fixed direction of the angular momentum with angular frequency

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

where ϕ is the Euler angle of the line of nodes with respect to the angular momentum as the space z axis.

- Using the results of Exercise 15, Chapter 4, show that ω rotates in space about the angular momentum with the same frequency $\dot{\phi}$, but that the angle θ' between ω and L is given by

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

where θ'' is the inclination of ω to the symmetry axis. Using the data given in Section 5.6, show therefore that Earth's rotation axis and axis of angular momentum are never more than 1.5 cm apart on Earth's surface.

- Show from parts (a) and (b) that the motion of the force-free symmetrical top can be described in terms of the rotation of a cone fixed in the body whose axis is the symmetry axis, rolling on a fixed cone in space whose axis is along the angular momentum. The angular velocity vector is along the line of contact of the two cones. Show that the same description follows immediately from the Poinsot construction in terms of the inertia ellipsoid.

Answer:

Marion shows that the angular momentum of the torque-free symmetrical top rotates in the body coordinates about the symmetry axis with an angular frequency ω more explicitly than Goldstein. Beginning with Euler's equation for force-free, symmetric, rigid body motion, we see that $\omega_3 = \textit{constant}$. The other Euler equations are

$$\dot{\omega}_1 = -\left(\frac{I_3 - I}{I}\omega_3\right)\omega_2$$

$$\dot{\omega}_2 = -\left(\frac{I_3 - I}{I}\omega_3\right)\omega_1$$

Solving these, and by already making the substitution, because we are dealing with constants,

$$\Omega = \frac{I_3 - I}{I}\omega_3$$

we get

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega(\omega_1 + i\omega_2) = 0$$

Let

$$q = \omega_1 + i\omega_2$$

Now

$$\dot{q} - i\Omega q = 0$$

has solution

$$q(t) = Ae^{i\Omega t}$$

this is

$$\omega_1 + i\omega_2 = A\cos\Omega t + iA\sin\Omega t$$

and we see

$$\omega_1(t) = A\cos\Omega t$$

$$\omega_2(t) = A\sin\Omega t$$

The x_3 axis is the symmetry axis of the body, so the angular velocity vector precesses about the body x_3 axis with a constant angular frequency

$$\Omega = \frac{I_3 - I}{I}\omega_3$$

.

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

To prove

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

We may look at the two cone figure angular momentum components, where L is directed along the vertical space axis and θ is the angle between the space and body vertical axis.

$$L_1 = 0$$

$$L_2 = L\sin\theta$$

$$L_3 = L\cos\theta$$

If α is the angle between ω and the vertical body axis, then

$$\omega_1 = 0$$
$$\omega_2 = \omega \sin\alpha$$
$$\omega_3 = \omega \cos\alpha$$

The angular momentum components in terms of α may be found

$$L_1 = I_1\omega_1 = 0$$
$$L_2 = I_1\omega_2 = I_1\omega \sin\alpha$$
$$L_3 = I_3\omega_3 = I_3\omega \cos\alpha$$

Using the Euler angles in the body frame, we may find, (using the instant in time where x_2 is in the plane of x_3, ω, and L, where $\psi = 0$) ,

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_2 = \dot{\phi}\sin\theta$$

This is

$$\dot{\phi} = \frac{\omega_2}{\sin\theta} = \frac{\omega\sin\alpha}{\sin\theta} = \omega(\frac{L_2}{I_1\omega})\frac{L}{L_2} = \frac{L}{I_1}$$

Plugging in L_3

$$\dot{\phi} = \frac{L}{I_1} = \frac{L_3}{I_1\cos\theta} = \frac{I_3\omega_3}{I_1\cos\theta}$$

A simple way to show

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

may be constructed by using the cross product of $\omega \times L$ and $\omega \times x_3$.

$$|\omega \times L| = \omega L \sin\theta' = L\sqrt{\omega_x^2 + \omega_y^2}$$

Using the angular velocity components in terms of Euler angles in the space fixed frame, this is equal to

$$\omega L \sin\theta' = L\dot{\phi}\sin\theta$$

with θ fixed, and $\dot{\theta} = 0$. For $\omega \times x_3$ we have

$$|\omega \times x_3| = \omega\sin\theta'' = \sqrt{\omega_{x'}^2 + \omega_{y'}^2}$$

Using the angular velocity components in terms of Euler angles in the body fixed frame, this is equal to

$$\omega \sin\theta'' = \dot{\phi}\sin\theta$$

Using these two expressions, we may find their ratio

$$\frac{\omega L \sin\theta'}{\omega \sin\theta''} = \frac{L\dot{\phi}\sin\theta}{\dot{\phi}\sin\theta}$$

$$\frac{\sin\theta'}{\sin\theta''} = \frac{\dot{\psi}}{\dot{\phi}}$$

Because $\dot{\psi} = \Omega$

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

To show that the Earth's rotation axis and axis of angular momentum are never more than 1.5cm apart on the Earth's surface, the following approximations may be made, $\sin\theta' \approx \theta'$, $\cos\theta \approx 1$, $\sin\theta'' \approx \theta''$, and $I_1/I_3 \approx 1$. Earth is considered an oblate spheroid, $I_3 > I_1$ and the data says there is $10m$ for amplitude of separation of pole from rotation axis. Using

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta''$$

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

we have

$$\sin\theta' = \frac{I_3 - I_1}{I_1}\omega_3\frac{I_1\cos\theta}{I_3\omega_3}\sin\theta''$$

Applying the approximations

$$\theta' = \frac{I_3 - I_1}{I_1}\theta''$$

$$\theta' = \frac{d}{R} = \frac{I_3 - I_1}{I_1}\frac{s}{R}$$

where R is the radius of the Earth, and s is the average distance of separation, which we will assume is half the amplitude, $5\,m$.

$$d = \frac{I_3 - I_1}{I_1}s = (.00327)(5) = 1.6\,cm$$

Force free motion means the angular momentum vector L is constant in time and stationary, as well as the rotational kinetic energy. (because the center of mass of the body is fixed). So because $T = \frac{1}{2}\omega \cdot L$ is constant, ω precesses around with a constant angle. This tracing is called the space cone, only if L is lined up with x_3 space axis. Proving that L, x_3 and ω all lie in the same plane will show that this space cone is traced out by ω. This results from $I_1 = I_2$ as shown below:

$$L \cdot (\omega \times e_3) = 0$$

because

$$\omega \times e_3 = \omega_2 e_1 - \omega_1 e_2$$

$$L \cdot (\omega \times e_3) = I_1 \omega_1 \omega_2 - I_2 \omega_1 \omega_2 = 0$$

Because $I_1 = I_2$.

Now the symmetry axis of the body has the angular velocity ω precessing around it with a constant angular frequency Ω. Thus another cone is traced out, the body cone. So we have two cones, hugging each other with ω in the direction of the line of contact.

5.7
For the general asymmetrical rigid body, verify analytically the stability theorem shown geometrically above on p. 204 by examining the solution of Euler's equations for small deviations from rotation about each of the principal axes. The direction of ω is assumed to differ so slightly from a principal axis that the component of ω along the axis can be taken as constant, while the product of components perpendicular to the axis can be neglected. Discuss the boundedness of the resultant motion for each of the three principal axes.

Answer:

Marion and Thornton give a clear analysis of the stability of a general rigid body. First lets define our object to have distinct principal moments of inertia. $I_1 < I_2 < I_3$. Lets examine the x_1 axis first. We have $\omega = \omega_1 e_1$ if we spin it around the x_1 axis. Apply some small perturbation and we get

$$\omega = \omega_1 e_1 + k e_2 + p e_3$$

In the problem, we are told to neglect the product of components perpendicular to the axis of rotation. This is because k and p are so small. The Euler equations

$$\begin{aligned} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) &= 0 \\ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) &= 0 \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) &= 0 \end{aligned}$$

become

$$I_1\dot{\omega}_1 - kp(I_2 - I_3) = 0$$
$$I_2\dot{k} - p\omega_1(I_3 - I_1) = 0$$
$$I_3\dot{p} - \omega_1 k(I_1 - I_2) = 0$$

Neglecting the product $pk \approx 0$, we see ω_1 is constant from the first equation. Solving the other two yields

$$\dot{k} = (\frac{I_3 - I_1}{I_2}\omega_1)p$$
$$\dot{p} = (\frac{I_1 - I_2}{I_3}\omega_1)k$$

To solve we may differentiate the first equation, and plug into the second:

$$\ddot{k} = (\frac{I_3 - I_1}{I_2}\omega_1)\dot{p} \quad \rightarrow \quad \ddot{k} + (\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^2)k = 0$$

Solve for $k(t)$:

$$k(t) = Ae^{i\Omega_{1k}t} + Be^{-i\Omega_{1k}t}$$

with

$$\Omega_{1k} = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$

Do this for $p(t)$ and you get

$$\Omega_{1k} = \Omega_{1p} \equiv \Omega_1$$

Cyclic permutation for the other axes yields

$$\Omega_1 = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$
$$\Omega_2 = \omega_2\sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_3 I_1}}$$
$$\Omega_3 = \omega_3\sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}$$

Note that the only unstable motion is about the x_2 axis, because $I_2 < I_3$ and we obtain a negative sign under the square root, Ω_2 is imaginary and the perturbation increases forever with time. Around the x_2 axis we have unbounded motion. Thus we conclude that only the largest and smallest moment of inertia rotations are stable, and the intermediate principal axis of rotation is unstable.

5.26
For the axially symmetric body precessing uniformly in the absence of torques, find the analytical solutions for the Euler angles as a function of time.

Answer:

For an axially symmetric body, symmetry axis L_z, we have $I_1 = I_2$, and Euler's equations are

$$I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3$$
$$I_3\dot{\omega}_3 = 0$$

This is equation (5.47) of Goldstein, only without the typos. Following Goldstein,

$$\omega_1 = A\cos\Omega t$$
$$\omega_2 = A\sin\Omega t$$

where

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

Using the Euler angles in the body fixed frame,

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

we have

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi = A\sin(\Omega t + \delta) \tag{1}$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = A\cos(\Omega t + \delta) \tag{2}$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} = constant \tag{3}$$

Multiplying the left hand side of (1) by $\cos\psi$ and the left hand side of (2) by $\sin\psi$, and subtracting them yields

$$[\dot{\phi}\sin\theta\sin\psi\cos\psi + \dot{\theta}\cos^2\psi] - [\dot{\phi}\sin\theta\cos\psi\sin\psi - \dot{\theta}\sin^2\psi] = \dot{\theta}$$

Thus we have

$$\dot{\theta} = A\sin(\Omega t + \delta)\cos\psi + A\cos(\Omega t + \delta)\sin\psi$$
$$\dot{\theta} = A\sin(\Omega t + \delta + \psi)$$

I assume uniform precession means $\dot{\theta} = 0$, no nutation or bobbing up and down. Thus

$$\Omega t + \delta + \psi = n\pi$$

with $n = 0, \pm 1, \pm 2...$, if $n = 0$

$$\psi = -\Omega t + \psi_0$$

where ψ_0 is the initial angle from the $x - axis$. From this, $\dot{\psi} = -\Omega$.

If we multiply the left hand side of (1) by $\sin\psi$ and the left hand side of (2) by $\cos\psi$, and add them:

$$[\dot{\phi}\sin\theta\sin^2\psi + \dot{\theta}\cos\psi\sin\psi] + [\dot{\phi}\sin\theta\cos^2\psi - \dot{\theta}\sin\psi\cos\psi] = \dot{\phi}\sin\theta$$

Thus we have

$$\dot{\phi}\sin\theta = A\sin(\Omega t + \delta)\sin\psi + A\cos(\Omega t + \delta)\cos\psi$$

$$\dot{\phi}\sin\theta = A\cos(\Omega t + \delta + \psi)$$

Plugging this result into (3)

$$\omega_3 = A\frac{\cos\theta}{\sin\theta}\cos(\Omega t + \psi + \delta) + \dot{\psi}$$

Using $\dot{\psi} = -\Omega$ and $\Omega t + \delta + \psi = 0$,

$$\omega_3 = A\frac{\cos\theta}{\sin\theta}\cos(0) - \Omega$$

$$A = (\omega_3 + \Omega)\tan\theta$$

and since $\Omega = \frac{I_3 - I_1}{I_1}\omega_3$

$$A = (\omega_3 + \frac{I_3 - I_1}{I_1}\omega_3)\tan\theta = \frac{I_3}{I_1}\omega_3\tan\theta$$

With this we can solve for the last Euler angle, ϕ,

$$\dot{\phi} = A\frac{\cos(\Omega t + \psi + \delta)}{\sin\theta} = \frac{I_3}{I_1}\omega_3\tan\theta\frac{\cos(0)}{\sin\theta}$$

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}$$

$$\phi = \frac{I_3\omega_3}{I_1\cos\theta}t + \phi_0$$

So all together

$$\theta = \theta_0$$

$$\psi(t) = -\Omega t + \psi_0$$

$$\phi(t) = \frac{I_3\omega_3}{I_1\cos\theta}t + \phi_0$$