Homework 11: # 10.7 b, 10.17, 10.26

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10.7

- A single particle moves in space under a conservative potential. Set up the Hamilton-Jacobi equation in ellipsoidal coordinates u, v, ϕ defined in terms of the usual cylindrical coordinates r, z, ϕ by the equations.

$$r = a \sinh v \sin u \qquad z = a \cosh v \cos u$$

For what forms of $V(u, v, \phi)$ is the equation separable.

- Use the results above to reduce to quadratures the problem of point particle of mass m moving in the gravitational field of two unequal mass points fixed on the z axis a distance $2a$ apart.

Answer:

Let's obtain the Hamilton Jacobi equation. This will be used to reduce the problem to quadratures. This is an old usage of the word quadratures, and means to just get the problem into a form where the only thing left to do is take an integral.

Here

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mr^2\dot{\phi}^2$$

$$r = a \sinh v \sin u$$

$$\dot{r} = a \cosh v \sin u \dot{v} + a \sinh v \cos u \dot{u}$$

$$z = a \cosh v \cos u$$

$$\dot{z} = a \sinh v \cos u \dot{v} - a \cosh v \sin u \dot{u}$$

Here

$$\dot{r}^2 + \dot{z}^2 = a^2(\cosh^2 v \sin^2 u + \sinh^2 v \cos^2 u)(\dot{v}^2 + \dot{u}^2) = a^2(\sin^2 u + \sinh^2 v)(\dot{v}^2 + \dot{u}^2)$$

To express in terms of momenta use

$$p_v = \frac{\partial L}{\partial \dot{v}} = ma^2(\sin^2 u + \sinh^2 v)\dot{v}$$

$$p_u = \frac{\partial L}{\partial \dot{u}} = ma^2(\sin^2 u + \sinh^2 v)\dot{u}$$

because the potential does not depend on $\dot{v}$ or $\dot{u}$. The cyclic coordinate ϕ yields a constant I'll call α_ϕ

$$p_\phi = mr^2\dot{\phi} = \alpha_\phi$$

So our Hamiltonian is

$$H = \frac{p_v^2 + p_u^2}{2ma^2(\sin^2 u + \sinh^2 v)} + \frac{p_\phi^2}{2ma^2 \sinh^2 v \sin^2 u} + V$$

To find our Hamilton Jacobi expression, the principle function applies

$$S = W_u + W_v + \alpha_\phi \phi - Et$$

So our Hamilton Jacobi equation is

$$\frac{1}{2ma^2(\sin^2 u + \sinh^2 v)}[(\frac{\partial W_u}{\partial u})^2 + (\frac{\partial W_v}{\partial v})^2] + \frac{1}{2ma^2 \sinh^2 v \sin^2 u}(\frac{\partial W_\phi}{\partial \phi})^2 + V(u, v, \phi) = E$$

This is

$$\frac{1}{2ma^2}[(\frac{\partial W_u}{\partial u})^2 + (\frac{\partial W_v}{\partial v})^2] + \frac{1}{2ma^2}(\frac{1}{\sinh^2 v} + \frac{1}{\sin^2 u})\alpha_\phi^2 + (\sin^2 u + \sinh^2 v)V(u, v, \phi) = (\sin^2 u + \sinh^2 v)E$$

A little bit more work is necessary. Once we solve for $V(u, v, \phi)$ we can then separate this equation into u, v and ϕ parts, at which point we will have only integrals to take.

I suggest drawing a picture, with two point masses on the z axis, with the origin being between them, so they are each a distance a from the origin. The potential is then formed from two pieces

$$V = -\frac{GmM_1}{|\vec{r} - a\hat{z}|} - \frac{GmM_2}{|\vec{r} + a\hat{z}|}$$

To solve for the denominators use the Pythagorean theorem, remembering we are in cylindrical coordinates,

$$|\vec{r} \mp a\hat{z}|^2 = (z \mp a)^2 + r^2$$

Using the results from part (a) for r and z,

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\cosh v \cos u \mp 1)^2 + a^2 \sinh^2 v \sin^2 u$$

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\cosh^2 v \cos^2 u \mp 2\cosh v \cos u + 1 + \sinh^2 v \sin^2 u)$$

Lets rearrange this to make it easy to see the next step,

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\sinh^2 v \sin^2 u + \cosh^2 v \cos^2 u + 1 \mp 2\cosh v \cos u)$$

Now convert the $\sin^2 u = 1 - \cos^2 u$ and convert the $\cosh^2 v = 1 + \sinh^2 v$

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\sinh^2 v + \cos^2 u + 1 \mp 2\cosh v \cos u)$$

Add the 1 and $\cosh^2 v$

$$|\vec{r} \mp a\hat{z}|^2 = a^2(\cosh^2 v + \cos^2 u \mp 2\cosh v \cos u)$$
$$|\vec{r} \mp a\hat{z}|^2 = (a(\cosh v \mp \cos u))^2$$

So our potential is now

$$V = -\frac{GmM_1}{a(\cosh v - \cos u)} - \frac{GmM_2}{a(\cosh v + \cos u)}$$

$$V = -\frac{1}{a}\frac{GmM_1(\cosh v + \cos u) + GmM_2(\cosh v - \cos u)}{\cosh^2 v - \cos^2 u}$$

Note the very helpful substitution

$$\cosh^2 v - \cos^2 u = \sin^2 u + \sinh^2 v$$

Allowing us to write V

$$V = -\frac{1}{a}\frac{GmM_1(\cosh v + \cos u) + GmM_2(\cosh v - \cos u)}{\sin^2 u + \sinh^2 v}$$

Plug this into our Hamilton Jacobi equation, and go ahead and separate out u and v terms, introducing another constant, A:

$$\frac{1}{2ma^2}(\frac{\partial W_u}{\partial u})^2 + \frac{1}{2ma^2}\frac{\alpha_\phi^2}{\sin^2 u} - \frac{1}{a}Gm(M_1 - M_2)\cos u - E\sin^2 u = A$$

$$\frac{1}{2ma^2}(\frac{\partial W_v}{\partial v})^2 + \frac{1}{2ma^2}\frac{\alpha_\phi^2}{\sinh^2 v} - \frac{1}{a}Gm(M_1 - M_2)\cosh v - E\sinh^2 v = -A$$

The problem has been reduced to quadratures.

10.17
Solve the problem of the motion of a point projectile in a vertical plane, using the Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time, assuming the projectile is fired off at time $t = 0$ from the origin with the velocity v_0, making an angle θ with the horizontal.

Answer:

I'm going to assume the angle is θ because there are too many α's in the problem to begin with. First we find the Hamiltonian,

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy$$

Following the examples in section 10.2, we set up the Hamiltonian-Jacobi equation by setting $p = \partial S/\partial q$ and we get

$$\frac{1}{2m}(\frac{\partial S}{\partial x})^2 + \frac{1}{2m}(\frac{\partial S}{\partial y})^2 + mgy + \frac{\partial S}{\partial t} = 0$$

The principle function is

$$S(x, \alpha_x, y, \alpha, t) = W_x(x, \alpha_x) + W_y(y, \alpha) - \alpha t$$

Because x is not in the Hamiltonian, it is cyclic, and a cyclic coordinate has the characteristic component $W_{q_i} = q_i \alpha_i$.

$$S(x, \alpha_x, y, \alpha, t) = x\alpha_x + W_y(y, \alpha) - \alpha t$$

Expressed in terms of the characteristic function, we get for our Hamiltonian-Jacobi equation

$$\frac{\alpha_x^2}{2m} + \frac{1}{2m}(\frac{\partial W_y}{\partial y})^2 + mgy = \alpha$$

This is

$$\frac{\partial W_y}{\partial y} = \sqrt{2m\alpha - \alpha_x^2 - 2m^2 gy}$$

Integrated, we have

$$W_y(y, \alpha) = -\frac{1}{3m^2 g}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{3/2}$$

Thus our principle function is

$$S(x, \alpha_x, y, \alpha, t) = x\alpha_x + -\frac{1}{3m^2 g}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{3/2} - \alpha t$$

Solving for the coordinates,

$$\beta = \frac{\partial S}{\partial \alpha} = -\frac{1}{mg}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{1/2} - t$$

$$\beta_x = \frac{\partial S}{\partial \alpha_x} = x + \frac{\alpha_x}{m^2 g}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{1/2}$$

Solving for both $x(t)$ and $y(t)$ in terms of the constants β, β_x, α and α_x

$$y(t) = -\frac{g}{2}(t+\beta)^2 + \frac{\alpha}{mg} - \frac{\alpha_x^2}{2m^2 g}$$

$$x(t) = \beta_x + \frac{\alpha_x}{m}(-\frac{1}{mg}(2m\alpha - \alpha_x^2 - 2m^2 gy)^{1/2})$$

Our $x(t)$ is

$$x(t) = \beta_x + \frac{\alpha_x}{m}(\beta + t)$$

We can solve for our constants in terms of our initial velocity, and angle θ through initial conditions,

$$x(0) = 0 \rightarrow \beta_x = -\frac{\alpha_x}{m}\beta$$

$$y(0) = 0 \rightarrow -\frac{g}{2}\beta^2 + \frac{\alpha}{mg} - \frac{\alpha_x^2}{2m^2 g} = 0$$

$$\dot{x}(0) = v_0 \cos\theta = \frac{\alpha_x}{m}$$

$$\dot{y}(0) = v_0 \sin\theta = -g\beta$$

Thus we have for our constants

$$\beta = \frac{v_0 \sin\theta}{-g}$$

$$\beta_x = \frac{v_0^2}{g}\cos\theta \sin\theta$$

$$\alpha = \frac{mg}{2g}(v_0^2 \sin^2\theta + v_0^2 \cos^2\theta) = \frac{mv_0^2}{2}$$

$$\alpha_x = mv_0 \cos\theta$$

Now our $y(t)$ is

$$y(t) = -\frac{g}{2}(t + \frac{v_0 \sin\theta}{g})^2 + \frac{v_0^2}{g} - \frac{v_0^2 \cos^2\theta}{2g}$$

$$y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta t - \frac{g}{2}\frac{v_0^2 \sin^2\theta}{g^2} + \frac{v_0^2}{g} - \frac{v_0^2 \cos^2\theta}{2g}$$

$$y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta t$$

and for $x(t)$

$$x(t) = \frac{v_0^2}{g}\cos\theta\sin\theta + v_0\cos\theta\frac{v_0\sin\theta}{-g} + v_0\cos\theta t$$

$$x(t) = v_0\cos\theta t$$

Together we have

$$y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta t$$

$$x(t) = v_0\cos\theta t$$

10.26
Set up the problem of the heavy symmetrical top, with one point fixed, in the Hamilton-Jacobi mehtod, and obtain the formal solution to the motion as given by Eq. (5.63).

Answer:

This is the form we are looking for.

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)^2}}$$

Expressing the Hamiltonian in terms of momenta like we did in the previous problem, we get

$$H = \frac{p_\psi^2}{2I_3} + \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$$

Setting up the principle function, noting the cyclic coordinates, we see

$$S(\theta, E, \psi, \alpha_\psi, \phi, \alpha_\phi, t) = W_\theta(\theta, E) + \psi\alpha_\psi + \phi\alpha_\phi - Et$$

Using

$$\frac{\partial S}{\partial q} = p$$

we have for our Hamilton-Jacobi equation, solved for the partial S's

$$\frac{\alpha_\psi^2}{2I_3} + \frac{1}{2I_1}(\frac{\partial W_\theta}{\partial\theta})^2 + \frac{(\alpha_\phi - \alpha_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta = E$$

Turning this inside out:

$$\frac{\partial}{\partial\theta}W_\theta(\theta,E) = \sqrt{2I_1E - \frac{\alpha_\psi^2 I_1}{I_3} - \frac{(\alpha_\phi - \alpha_\psi\cos\theta)^2}{\sin^2\theta} - 2I_1Mgh\cos\theta}$$

When integrated,

$$W_\theta = \int (2I_1E - \frac{\alpha_\psi^2 I_1}{I_3} - \frac{(\alpha_\phi - \alpha_\psi\cos\theta)^2}{\sin^2\theta} - 2I_1Mgh\cos\theta)^{1/2} d\theta$$

Now we are in a position to solve

$$\beta_\theta = \frac{\partial S}{\partial E} = \frac{\partial W_\theta}{\partial E} - t$$

$$\frac{\partial W_\theta}{\partial E} = \beta_\theta + t = \int \frac{2I_1 d\theta}{2(2I_1E - \frac{\alpha_\psi^2 I_1}{I_3} - \frac{(\alpha_\phi - \alpha_\psi\cos\theta)^2}{\sin^2\theta} - 2I_1Mgh\cos\theta)^{1/2}}$$

Using the same constants Goldstein uses

$$\alpha = \frac{2E - \frac{\alpha_\psi^2}{I_3}}{I_1} = \frac{2E}{I_1} - \frac{\alpha_\psi^2}{I_3 I_1}$$
$$\beta = \frac{2Mgl}{I_1}$$

where

$$\alpha_\phi = I_1 b$$
$$\alpha_\psi = I_1 a$$

and making these substitutions

$$\beta_\theta + t = \int \frac{I_1 d\theta}{(I_1(2E - \frac{\alpha_\psi^2}{I_3}) - I_1^2\frac{(b - a\cos\theta)^2}{\sin^2\theta} - I_1 2Mgh\cos\theta)^{1/2}}$$
$$\beta_\theta + t = \int \frac{d\theta}{(\alpha - \frac{(b - a\cos\theta)^2}{\sin^2\theta} - \beta\cos\theta)^{1/2}}$$

For time t, the value of θ is $\theta(t)$

$$t = \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{(\alpha - \frac{(b - a\cos\theta)^2}{\sin^2\theta} - \beta\cos\theta)^{1/2}}$$

The integrand is the exact expression as Goldstein's (5.62). Making the substitution $u = \cos\theta$ we arrive home

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)^2}}$$