Homework 1: # 1.4, 1.5, 1.6, 1.10, 1.12, 1.13

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Problem 1.4

Each of three charged spheres of radius a one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n $(n > -3)$, has a total charge Q. Use Gauss's theorem to obtain the elctric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.

Solution:

For a conducting sphere, charge resides on the surface. Even in an external electric field induced charges on the surface will produce a field of their own and cancel off the original field. The net electric field inside a conductor is always zero.

$$r < a \quad \oint_S E \cdot \hat{r}\ da = \frac{1}{\epsilon_0}\int_V \rho(r) dV \quad \rho(r) = 0 \rightarrow \vec{E} = 0$$

Outside the conductor there is no cancellation of the electric field.

$$r > a \quad \oint_S E \cdot \hat{r}\ da = \frac{Q}{\epsilon_0}$$

because

$$Q = \int_V \rho(r) dV$$

we then have

$$E \cdot (Area) = \frac{Q}{\epsilon_0}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

for outside the conducting sphere, $r > a$. And actually, this is the case for outside all three spheres, as outside, the total charge for all three are the same Q.

The electric field inside a sphere of uniform charge density is found by Gauss's law

$$r < a \quad E(4\pi r^2) = \frac{1}{\epsilon_0}\int_V \rho(r) dV$$

$$E = \frac{1}{\epsilon_0}\frac{1}{4\pi r^2}\frac{3Q}{4\pi a^3}\frac{4}{3}\pi r^3$$

$$r < a \quad \vec{E} = \frac{Qr}{4\pi\epsilon_0 a^3}\hat{r}$$

for inside a sphere of uniform charge density.

The electric field inside a sphere having spherically symmetric charge density varying radially as $r^n (n > -3)$ is found by

$$r < a \quad E \cdot 4\pi r^2 = \frac{1}{\epsilon_0}\int kr^n dV$$

$$E \cdot 4\pi r^2 = \frac{4\pi k}{\epsilon_0}\int r^{n+2} dr$$

$$E = \frac{kr^{n+1}}{\epsilon_0(n+3)}\hat{r}$$

To find k, we use

$$Q = \int_0^a \rho(r) dV$$

$$Q = 4\pi k \int_0^a r^{n+2} dr$$

$$k = \frac{Q(n+3)}{4\pi a^{n+3}}$$

Thus

$$r < a \quad \vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{r^{n+1}}{a^{n+3}}\hat{r}$$

for inside a sphere of spherically symmetric charge density that varies radially as r^n $(n > -3)$.

Conducting sphere sketch:

Uniform density sphere sketch:

Symmetric charge density sketch, $n = -2$ and $n = 2$:

Problem 1.5

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon}\frac{e^{-\alpha r}}{r}(1 + \frac{\alpha r}{2})$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge(both continuous and discrete) that will give this potential and interpret your result physically.

Solution:

Using Poisson's equation,

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

we can solve for the charge density, which is the distribution of charge. We are given a hint to find the discrete distribution of charge, meaning likely a delta function will be in our answer. In spherical coordinates, ∇^2 is defined by

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$$

Also, Jackson shows on page 35, that

$$\nabla^2(\frac{1}{r}) = -4\pi\delta(\vec{r})$$

Now lets solve Poisson's equation for ρ and see if this gives us something that might make sense for the hydrogen atom.

$$\nabla^2\Phi = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})[\frac{e^{-\alpha r}}{r} + \frac{\alpha e^{-\alpha r}}{2}]$$

Using the product rule on the first term, and setting $\nabla^2\Phi = -\rho/\epsilon_0$ we obtain,

$$\rho = -\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}(e^{-\alpha r}r^2\frac{\partial}{\partial r}(\frac{1}{r}) - \alpha r e^{-\alpha r} - \frac{\alpha^2 r^2}{2}e^{-\alpha r})$$

It only gets slightly more messy from here. If we distribute the $\frac{1}{r^2}\frac{\partial}{\partial r}$ term we get

$$\rho = -\frac{q}{4\pi}[\frac{1}{r^2}\frac{\partial}{\partial r}(e^{-\alpha r}r^2\frac{\partial}{\partial r}(\frac{1}{r})) - \frac{1}{r^2}\frac{\partial}{\partial r}(\alpha r e^{-\alpha r}) - \frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\alpha^2 r^2}{2}e^{-\alpha r})]$$

The product rule gives us 6 terms from this.

$$\rho = -\frac{q}{4\pi}[e^{-\alpha r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})(\frac{1}{r}) + \frac{\alpha}{r^2}e^{-\alpha r} + \frac{\alpha^2}{r}e^{-\alpha r} - \frac{\alpha}{r^2}e^{-\alpha r} + \frac{\alpha^3}{2}e^{-\alpha r} - \frac{\alpha^2}{r}e^{-\alpha r}]$$

This is, after the terms cancel,

$$\rho = -\frac{q}{4\pi}[e^{-\alpha r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})(\frac{1}{r}) + \frac{\alpha^3}{2}e^{-\alpha r}]$$

Using our delta function equation for the first term

$$\rho = -\frac{q}{4\pi}[-4\pi\delta(\vec{r}) + \frac{\alpha^3}{2}e^{-\alpha r}]$$

$$\rho = q\delta(\vec{r}) - \frac{q}{8\pi}\alpha^3 e^{-\alpha r}$$

Physically, this is the point charge of the proton nucleus represented by the delta function at the center of the atom, surrounded by the negative electron cloud.

Problem 1.6

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of

- two large, flat, conducting sheets of area A, separated by a small distance d;
- two concentric conducting spheres with radii a,b ($b > a$);
- two concentric cnoducting cylinders of length L, large compared to their radii a, b, ($b > a$).
- What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylinderical wire of diameter 1mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12}F/m?

Solution:

For two conducting sheets, Griffiths (pg 105) does a fine job explaining what happens. Gauss's law is used to find the electric field.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\int \vec{E} \cdot d\vec{a} = 2A|\vec{E}|$$

$$2A|\vec{E}| = \frac{\sigma A}{\epsilon_0}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

The electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0}$$

as the fields cancel outside the plates, but contribute inside. We are looking for capacitance,

$$C = \frac{Q}{V}$$

and the potential difference is

$$V = \int_0^d \vec{E} \cdot d\vec{l} = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{\epsilon_0 A}$$

Thus

$$C = \frac{A\epsilon_0}{d}$$

For two conducting spheres, Griffiths ex 2.11 explicitly derives the capacitance. We know from Gauss's law the electric field between the two shell's from the problem I just did, Jackson 1.4.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

So we only need the potential, because $C = Q/V$.

$$V = -\int_b^a \vec{E} \cdot d\vec{l} = -\frac{Q}{4\pi\epsilon_0}\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\epsilon_0}(\frac{1}{a} - \frac{1}{b})$$

Therefore

$$C = \frac{Q}{V} = 4\pi\epsilon_0\frac{ab}{b-a}$$

For two conducting cylinders, we use Gauss's law to find the electric field

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0}$$

$$|\vec{E}|2\pi lL = \frac{Q}{\epsilon_0}$$

$$\vec{E} = \frac{Q}{2\pi lL\epsilon_0}\hat{s}$$

Finding the potential difference as before,

$$V = -\int_b^a \vec{E} \cdot d\vec{l} = -\frac{Q}{2\pi L\epsilon_0}\int_b^a \frac{1}{l}dl = \frac{Q}{2\pi L\epsilon_0}\ln\frac{b}{a}$$

Therefore,

$$C = \frac{Q}{V} = \frac{2\pi L\epsilon_0}{\ln\frac{b}{a}}$$

For part (d) we just use the above formula and plug and chug into a calculator, making the appropriate unit conversions.

$$\epsilon_0 = 8.85 \times 10^{-12} C^2/Nm^2$$

$$2\pi\epsilon_0 = 3 \times 10^{-11} \ln\frac{b}{.5 \times 10^{-3}}$$

$$b = 3.2 \times 10^{-3}$$

The diameter is twice this, so our first answer is

$$d_1 = 6.4 \times 10^{-3}\ m$$

For $3 \times 10^{-12} F/m$ we do the same

$$2\pi\epsilon_0 = 3 \times 10^{-12} \ln \frac{b}{.5 \times 10^{-3}}$$

and we get

$$d_2 = 1.1 \times 10^{5}\ m$$

Together for part (d) the answers are

$$d_1 = 6.4 \times 10^{-3}\ m \quad d_2 = 1.1 \times 10^{5}\ m$$

This is a big difference for only a magnitude difference in capacitance.

Problem 1.10

Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

Solution:

What we are looking to prove mathematically is this statement:

$$\Phi(x) = \frac{1}{4\pi R^2} \oint \Phi(x') d^3x'$$

This is the potential at any point equal to the average of the potential over the surface of any sphere. Section 1.8 of Jackson is most helpful for proving this theorem. Taking Jackson's lead and noting the comments made toward the end of the section about charge-free volume, lets start with Green's Theorem:

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi) d^3x = \oint_S [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}] da$$

He does a few things to this equation. Namely, he sets

$$\psi = \frac{1}{R} \quad \phi = \Phi$$

We will do the same. We will also use x' to be the integration variable. Lets look at the four integrals and see what we can do with them.

$$\int_V \Phi(x')\nabla^2(\frac{1}{R}) d^3x' = \int_V \Phi(x')[-4\pi\delta(x - x')] d^3x' = -4\pi\Phi(x)$$

because $\nabla^2(1/R) = -4\pi\delta(x - x')$. Take a look at the second integral

$$\int_V -\frac{1}{R}\nabla^2\Phi(x')d^3x' = \int_V \frac{1}{R}\frac{\rho}{\epsilon_0}d^3x' = 0$$

because there is no charge in the volume we are integrating. Charge-free volume. The third integral:

$$\oint_S \Phi(x')\frac{\partial}{\partial n}(\frac{1}{R})d^2x' = -\oint_S \Phi(x')\frac{1}{R^2}d^3x'$$

This is looking familiar, and we should feel on the right track. But what about the fourth integral?

$$\oint_S -\frac{1}{R}\frac{\partial\Phi(x')}{\partial n}d^2x' = \oint_S -\frac{1}{R}(\nabla\Phi(x')\cdot\hat{n}')d^2x' = \oint_S \frac{1}{R}(\vec{E}\cdot\hat{n}')d^2x'$$

Using the divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{A}\,d^3x = \oint_S \vec{A}\cdot\vec{n}\,da$$

we may change the fourth integral into

$$\oint \frac{1}{R}\vec{E}\cdot\hat{n}'d^2x' = \frac{1}{R}\int_V \vec{\nabla}\cdot\vec{E}d^3x' = \frac{1}{R}\int_V \frac{\rho}{\epsilon_0}d^3x' = 0$$

because again, we are in a charge-free volume. So we are left with only the first and third integrals,

$$-4\pi\Phi(x) = -\oint_S \Phi(x')\frac{1}{R^2}d^3x'$$

and voila, the mean value theorem for electrostatic potential in charge-free space:

$$\Phi(x) = \frac{1}{4\pi R^2}\oint_S \Phi(x')d^3x'$$

Problem 1.12

Prove *Green's reciprocation theorem*: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi'd^3x + \int_S \sigma\Phi'da = \int_V \rho'\Phi d^3x + \int_S \sigma'\Phi da$$

Solution:

Using Green's theorem and replacing

$$\psi \rightarrow \Phi' \qquad \phi \rightarrow \Phi$$

and not forgetting equation (1.28)

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \qquad \nabla^2\Phi' = -\frac{\rho'}{\epsilon_0}$$

and remembering the interpretation of the normal derivative of the potential derived from boundary conditions to yeild a surface-charge density, as explained most elegantly in section 2.3.5 in Griffiths

$$\sigma = \epsilon_0\frac{\partial\Phi}{\partial n} \qquad \sigma' = \epsilon_0\frac{\partial\Phi'}{\partial n}$$

we have

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_S [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}]da$$

go ahead and replace ψ and ϕ and rearrange terms:

$$-\int_V \Phi'\nabla^2\Phi d^3x + \oint_S \Phi'\frac{\partial\Phi}{\partial n}da = -\int_V \Phi\nabla^2\Phi' d^3x + \oint_S \Phi\frac{\partial\Phi'}{\partial n}da$$

plugging in, we get

$$\int_V \Phi'\frac{\rho}{\epsilon_0}d^3x + \oint_S \Phi'\frac{\sigma}{\epsilon_0}da = \int_V \Phi\frac{\rho'}{\epsilon_0}d^3x + \oint_S \Phi\frac{\sigma'}{\epsilon_0}da$$

Cancel out the ϵ_0's and voila, Green's reciprocation theorem:

$$\int_V \rho\Phi' d^3x + \int_S \sigma\Phi' da = \int_V \rho'\Phi d^3x + \int_S \sigma'\Phi da$$

Problem 1.13

Two infinite grounded parallel conducting planes are separated by a distance d. A point charge q is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to (-q) times the fractional perpendicular distance of the point charge from the other plane. (HINT: As your comparison electrostatic problem with the same surfaces choose one whose charge densities and potential are known and simple.)

Solution:

We want to use Green's reciprocation theorem:

$$\int_V \rho\Phi' d^3x + \int_S \sigma\Phi' da = \int_V \rho'\Phi d^3x + \int_S \sigma'\Phi da$$

to prove the top plate has a charge

$$Q_{top} = -q\frac{l}{d}$$

where l is the distance from the bottom plate, in the z direction. The hint is trying to get us to use the electrostatic potential for a parallel plate capacitor. That is

$$\Phi' = V\frac{z}{d}$$

Where V is the potential of the top plate, and z is the distance from the bottom plate. As we may plug in, we can see this works for

$$z = 0 \rightarrow \Phi'_{bot} = 0$$

$$z = d \rightarrow \Phi'_{top} = V$$

For the unprimed case with the charge in the middle, Jackson equation (1.6) gives us the charge density by means of a delta function. Also, the potentials vanish because the plates are grounded.

$$\rho(\vec{x}) = \sum_{i=1} q_i\delta(\vec{x} - \vec{x_i}) = q\delta x\delta y\delta z - l$$

$$\Phi_{top} = \Phi_{bot} = 0$$

So for the primed case with no charge in the middle we have together:

$$\rho' = 0$$

$$\Phi' = V\frac{z}{d}$$

Plugging the $\Phi = 0$ into Green's reciprocation theorem for the surface integral and $\rho' = 0$ we get:

$$\int_V \rho\Phi' d^3x + \oint_S \sigma\Phi' d^2x = 0$$

Plugging ρ and separating the surface integral for the two plates yields

$$\int_V q\delta(x)\delta(y)\delta(z-l)\Phi' d^3x + \oint_{S_{bot}} \sigma_{bot}\Phi'_{bot} d^2x + \oint_{S_{top}} \sigma_{top}\Phi'_{top} d^2x = 0$$

Finally, plugging in $\Phi' = Vz/d$:

$$\int_V q\delta(x)\delta(y)\delta(z-l)V\frac{z}{d}d^3x + \oint_{S_{bot}} \sigma_{bot}[0]d^2x + \oint_{S_{top}} \sigma_{top}[V]d^2x = 0$$

This is

$$Vq\frac{l}{d} + 0 + V\oint_{S_{top}} \sigma_{top}d^2x = 0$$

$$Vq\frac{l}{d} + VQ_{top} = 0$$

So we have

$$Q_{top} = -q\frac{l}{d}$$