Homework 2: # 2.2, 2.5, 2.7, 2.11

Michael Good

Jan 20, 2005

Problem 2.2

Using the m[e]thod of images, discuss the problem of a point charge q inside a hollow, grounded, conducting sphere of inner radius a. Find

- the potential inside the sphere
- the induced surface-charge density
- the magnitude and direction of the force acting on q
- Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

Solution:

a. We will have the same results as if the charge was outside the sphere. Equation 2.2:

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right]$$

Equation (2.3) and (2.4) show that to meet the boundary condition $\phi(x = a) = 0$ we have

$$q' = -\frac{a}{y}q \quad y' = \frac{a^2}{y}$$

therefore the potential is:

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|x\hat{n} - y\hat{n}'|} - \frac{qa}{y|x\hat{n} - \frac{a^2}{y}\hat{n}'|}\right]$$

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|x\hat{n} - y\hat{n}'|} - \frac{qa}{|yx\hat{n} - a^2\hat{n}'|}\right]$$

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2-2xy\cos\gamma}} - \frac{qa}{\sqrt{x^2y^2+a^4-2xya^2\cos\gamma}}\right]$$

Where I proceeded the same way Jackson did in Section 2.2.

b. For the surface charge density, the normal derivative out of the conductor is now radially inward, meaning everything is exactly the same as before except we have a change in sign.

$$\sigma_{outside} = -\epsilon_0\frac{\partial\phi}{\partial\hat{n}} \qquad \sigma_{inside} = \epsilon_0\frac{\partial\phi}{\partial\hat{n}}$$

Thus

$$\sigma_{inside} = \frac{q}{4\pi a^2}(\frac{a}{y})\frac{1-\frac{a^2}{y^2}}{(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma)^{\frac{3}{2}}}$$

that is, equation 2.5 with a sign change.

c. Here the distance is now

$$y' - y = \frac{a^2}{y} - y = y(\frac{a^2}{y^2}-1)$$

So

$$|F| = \frac{1}{4\pi\epsilon_0}\frac{q^2a}{y^3(\frac{a^2}{y^2}-1)^2}$$

$$|F| = \frac{q^2a}{4\pi\epsilon_0}\frac{y}{(y^2)^2(\frac{a^2}{y^2}-1)^2} = \frac{q^2a}{4\pi\epsilon_0}\frac{y}{(a^2-y^2)^2}$$

so

$$\vec{F} = \frac{q^2a}{4\pi\epsilon_0}\frac{y}{(a^2-y^2)^2}\hat{y}$$

where the force is directed outward along y.

d. Now is there any change if the sphere is kept at V? The potential is

$$\phi = \phi_{grd} + V$$

This satisfies the boundary conditions of internal charge distribution and potential on boundary. Thus because they are a solution to Poisson's Equ. and the uniqueness theorem holds, this is the only solution. The charge surface density stays the same. (there are no charges on the surface with just a sphere kept at V). The electric fields are the same so the force does not change.

Any change if the sphere has a total charge Q on its inner and outer surface?

$$Q = Q_I + Q_O$$

$$Q = -q + Q_O$$

for no charge inside, charge resides on surface of conductor.

$$Q + q = Q_0$$

$$V = \frac{Q_O}{4\pi\epsilon_0 b} \qquad a \le x \le b$$

where b is the outer radius. So, if we were inside the inner radius

$$\phi = \phi_{grd} + \frac{Q_O}{4\pi\epsilon_0 b} \qquad x \le a$$

The surface charge density and forces stays the same.

Problem 2.5

- Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, Eq. (2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)}$$

  Relate this result to the electrostatic potential, Eq. (2.3), and the energy discussion of Section 1.11.

- Repeat the calculation of the work done to remove the charge q against the force, Eq. (2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

  Relate the work to the electrostatic potential, Eq. (2.8), and the energy discussion of Section 1.11.

Solution:

a. The equation is

$$F = \frac{q^2 a}{4\pi\epsilon_0} \frac{y}{y^4(1 - \frac{a^2}{y^2})^2} = \frac{q^2 a y}{4\pi\epsilon_0 (y^2 - a^2)^2}$$

So lets find the work:

$$W = \int_r^\infty F dy = \int_r^\infty \frac{q^2 a y}{4\pi\epsilon_0 (y^2 - a^2)^2} dy = \frac{q^2 a}{4\pi\epsilon_0} \int \frac{du}{2u^2}$$

with $u = y^2 - a^2$ and $du = 2ydy$.

$$W = -\frac{q^2 a}{8\pi\epsilon_0 (y^2 - a^2)}\bigg|_r^\infty = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)}$$

Finding the potential energy, using the potential, we see that:

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq'}{|r - r'|} = \frac{1}{4\pi\epsilon_0}\frac{-q^2 a}{r(r - \frac{a^2}{r})} = -\frac{q^2 a}{4\pi\epsilon_0 (r^2 - a^2)}$$

using $q' = -qa/r$ and $r' = a^2/r$.

Why in the world is it off by a factor of two?? Its because in the first case we are using image charges, which move and increase in charge magnitude, while for the second case the potential energy term is calculated assuming there is no movement for the second charge and no change in its magnitude.

b. Equation 2.9 may be thought of as the total force of two image charges, one at the origin with magnitude $Q - q'$ and one in the same place with the same charge as above. It's most easy to just add the work from above to the work needed to remove our point charge to infinity from the charge at the origin. The total work is then:

$$W = W_a + W_0 = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)} - \int_r^\infty \frac{q(Q - q')}{4\pi\epsilon_0 y^2} dy$$

$$W = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)} - \int_r^\infty \frac{qQ}{4\pi\epsilon_0 y^2} dy + \int_r^\infty q\frac{qa}{y}\frac{1}{4\pi\epsilon_0 y^2} dy$$

$$W = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)} - \frac{qQ}{4\pi\epsilon_0 r} - \frac{q^2 a}{2 \cdot 4\pi\epsilon_0 r^2}$$

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{qQ}{r} - \frac{q^2 a}{2r^2}\right]$$

For the potential energy we use the same U from part (a) and add to the additional potential energy caused by the charge at the origin.

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{(r^2 - a^2)} - \frac{q(Q + \frac{a}{r}q)}{r}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{(r^2 - a^2)} - \frac{qQ}{r} - \frac{q^2 a}{r^2}\right]$$

So here's our factors of 2 again. They don't appear because we are not dealing with image charges. But the middle term is the same because he is fixed on the sphere.

Problem 2.7

Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

- Write down the appropriate Green function $G(\vec{x}, \vec{x}')$.
- If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ψ, z).
- Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

- Show that at large distances ($\rho^2 + z^2 >> a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + ...\right]$$

  Verify that the results of parts c and d are consistent with each other in their common range of validity.

Solution:

a. The Green Function for the half-space is:

$$G(r, r') = \frac{1}{|r - r'|} - \frac{1}{|r - r''|}$$

$$G(r, r') = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}$$

b. Using equation Jackson (1.44), and recognizing there is no free charges:

$$\phi = -\frac{1}{4\pi}\oint \phi(x')\frac{\partial G}{\partial n'}da'$$

It is standard notation for n' to point out of the volume you're in. We are above the plane where $z \geq 0$ so n' points along $-z$.

$$\frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial z'}$$

So taking the partial

$$\frac{\partial G}{\partial z'} = \frac{-\frac{1}{2}(2)(-)(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} - \frac{-\frac{1}{2}(2)(z+z')}{((x-x')^2+(y-y')^2+(z+z')^2)^{3/2}}$$

Plugging in $z' = 0$ because we are using a surface integral. And surface integrals are evaluated at the surface!

$$\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = \frac{2z}{((x-x')^2+(y-y')^2+(z)^2)^{3/2}}$$

Now the two negatives cancel, the one with the integral and the one with the partial, so we are left with:

$$\phi = \frac{1}{4\pi}\oint V \frac{2z}{((x-x')^2+(y-y')^2+(z)^2)^{3/2}}$$

where I used $\phi(x') = V$ inside the circle, otherwise it'd be zero, and I'd have no answer. Converting to cylindrical coordinates

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

we have an integral expression for the potential

$$\phi = \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^a d\rho' V \frac{2z\rho'}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{3/2}}$$

c. We know $\rho = 0$ along the axis so

$$\phi(0,\phi,z) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^a dp' \frac{Vzp'}{(\rho'^2+z^2)^{3/2}}$$

$$\phi(0,\phi,z) = \int_0^a dp' \frac{Vzp'}{(\rho'^2+z^2)^{3/2}}$$

Perform a u-substitution for this integral. $u = (\rho'^2 + z^2)$ and $du = 2\rho' d\rho'$

$$\int \frac{zV du}{2u^{3/2}} = -\frac{zV}{u^{1/2}} = -\left.\frac{zV}{\sqrt{\rho'^2+z^2}}\right|_0^a$$

$$\phi = -\frac{zV}{\sqrt{a^2+z^2}} + \frac{zV}{z}$$

$$\phi = V(1 - \frac{z}{\sqrt{a^2+z^2}})$$

d. This part is tedious algebraically. First I expressed the potential like so:

$$\phi = \frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' \int_0^a d\rho'\rho'(1 + \frac{\rho'^2 - 2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2})^{-3/2}$$

Now in a position to start expanding, the binomial theorem is:

$$(1+x)^{-3/2} = 1 + [-\frac{3}{2}x] + \frac{15}{8}x^2 - ...$$

Hopefully it's obvious what my x is. It's the huge term full of ρ's in the integral. I'm going to go ahead and integrate term by term of the expansion, starting with 1:

$$\int_0^{2\pi}\int_0^a \rho' d\rho' d\phi' = \pi a^2$$

next with $-\frac{3}{2}x$:

$$-\frac{3}{2}\int_0^{2\pi}\int_0^a \frac{\rho'^2 - 2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\rho' d\rho' d\phi' = -\frac{3\pi a^4}{4(\rho^2+z^2)}$$

and last term, $\frac{15}{8}x^2$:

$$\frac{15}{8}\int_0^{2\pi}\int_0^a \frac{\rho'^4 - 4\rho\rho'^3\cos(\phi-\phi') + 4\rho^2\rho'^2\cos^2(\phi-\phi')}{(\rho^2+z^2)^2}\rho' d\rho' d\phi' = \frac{5\pi a^6}{8(\rho^2+z^2)^2} + \frac{15\pi\rho^2 a^4}{8(\rho^2+z^2)^2}$$

This was done using the integral: $\int_0^{2\pi}\cos^2(\phi-\phi')d\phi' = \pi$. If you add up the terms and multiply by the term outside the integral, the overall potential is

$$\phi(\rho,z) = \frac{Va^2 z}{2(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5a^2}{8(\rho^2+z^2)^2} + \frac{15a^2\rho^2}{8(\rho^2+z^2)^2} + ...)\right]$$

These are the leading terms the problem asks to produce. With this answer, I can now show that along the axis for part (d) and at large distances for part (c) the terms should match because this is their common range of validity.

- Part (d) result along the axis :

$$\phi(\rho=0,z) = \frac{Va^2}{2z^2}\left[1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4}...\right]$$

- Part (c) result for large distances, z:

$$\phi(\rho=0,z) = V(1 - \frac{z}{\sqrt{a^2+z^2}}) = V(1 - \frac{1}{\sqrt{1+\frac{a^2}{z^2}}}) \approx \frac{Va^2}{2z^2}\left[1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4}...\right]$$

using the binomial theorem.

Problem 2.11

A line charge with linear charge density τ is placed parallel to, and a distance R away from, the axis of a conducting cylinder of radius b held at fixed voltage such that the potential vanishes at infinity. Find

(a) the magnitude and position of the image charge(s);

(b) the potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x axis), including the asymptotic form far from the cylinder;

(c) the induced surface-charge density, and plot it as a function of angle for $R/b = 2, 4$ in units of $\tau/2\pi b$;

(d) the force per unit length on the line charge.

Solution:

a. The potential of a line charge is:

$$\phi = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{r^2}$$

Here, R is a constant, and $r^2 = (x - x_0)^2 + (y - y_0)^2$. Since we have a linear charge density τ lets set the image linear charge density to τ'. So our situation with two lines is now

$$\phi = \frac{\tau}{4\pi\epsilon_0}\ln\frac{R^2}{r^2} + \frac{\tau'}{4\pi\epsilon_0}\ln\frac{R^2}{r'^2}$$

The image line charge will be inside the cylinder and will have the opposite but same magnitude charge density, i.e. $\tau = -\tau'$. This means that if we are super far away there will be a zero potential. So our potential is

$$\phi = \frac{\tau}{4\pi\epsilon_0}\ln\frac{R^2}{r^2} - \frac{\tau}{4\pi\epsilon_0}\ln\frac{R^2}{r'^2}$$

$$\phi = \frac{\tau}{4\pi\epsilon_0}\ln\frac{r'^2}{r^2}$$

But if we come real close to the cylinder, right up to the surface, we want the potential to be fixed. (like it states in the problem) So our expression for ϕ must be made so

$$\phi(r = b) = \phi(r = -b)$$

This gives us

$$\phi = \frac{\tau}{2\pi\epsilon_0} \ln \frac{|r - R'|}{|r - R|}$$

Where R' acts as x_0' the position of the image line, and R acts as x_0 the position of the real line charge.

$$\phi(b) = \phi(-b) \rightarrow \ln \frac{|b - R'|}{|b - R|} = \ln \frac{|(-b) - R'|}{|(-b) - R|}$$

Since R is greater than b we have

$$\ln \frac{b - R'}{R - b} = \ln \frac{(-b) - R'}{(-b) - R}$$

$$\frac{b - R'}{R - b} = \frac{b + R'}{b + R}$$

Where b, R, and R' are still vectors. This is:

$$(b + R)(b - R') = (R - b)(b + R')$$

$$b^2 - b \cdot R - RR' = RR' - b^2 - b \cdot R'$$

The angle does not matter because wherever we are on the surface of the cylinder we will still have a fixed V. So the dot terms cancel. We are left with

$$2b^2 = 2RR' \qquad \rightarrow \qquad R' = \frac{b^2}{R}$$

This is the same as the sphere except the potential at the surface of the cylinder is V where as the sphere is *zero*.

b. Converting coordinates:

$$\phi = \frac{\tau}{4\pi\epsilon_0} \ln \frac{\rho^2 + R'^2 - 2\rho R' \cos\phi}{\rho^2 + R^2 - 2\rho R \cos\phi}$$

If we want the asymptotic form far from the cylinder let me divide by ρ and then plug in a huge ρ compared to R or R':

$$\phi = \frac{\tau}{4\pi\epsilon_0} \ln \frac{1 - 2\frac{R'}{\rho} \cos\phi}{1 - 2\frac{R}{\rho} \cos\phi}$$

$$\phi = \frac{\tau}{4\pi\epsilon_0} (\ln[1 - 2\frac{R'}{\rho} \cos\phi] - \ln[1 - 2\frac{R}{\rho} \cos\phi])$$

Using $\ln[1 + x] = x$ when x is small:

$$\phi = \frac{\tau}{4\pi\epsilon_0} (-2\frac{R'}{\rho} \cos\phi + 2\frac{R}{\rho} \cos\phi)$$

$$\phi = \frac{\tau}{4\pi\epsilon_0}[\frac{2}{\rho}(R - R')\cos\phi]$$

Plugging in our result from part (a):

$$\phi = \frac{\tau}{4\pi\epsilon_0}[\frac{2}{\rho}(R - \frac{b^2}{R})\cos\phi]$$

c. The induced surface charge density is found from:

$$\sigma = -\,\epsilon_0 \frac{\partial\phi}{\partial\rho}\bigg|_{\rho=b}$$

Using

$$\phi = \frac{\tau}{4\pi\epsilon_0}\ln\frac{\rho^2 + R'^2 - 2\rho R'\cos\phi}{\rho^2 + R^2 - 2\rho R\cos\phi}$$

$$\phi = \frac{\tau}{4\pi\epsilon_0}\left[\ln(\rho^2 + R'^2 - 2\rho R'\cos\phi) - \ln(\rho^2 + R^2 - 2\rho R\cos\phi)\right]$$

Lets take a derivative, remembering the chain rule:

$$\sigma = -\,\epsilon_0 \frac{\tau}{4\pi\epsilon_0}\left(\frac{2\rho - 2R'\cos\phi}{\rho^2 + R'^2 - 2\rho R'\cos\phi} - \frac{2\rho - 2R\cos\phi}{\rho^2 + R^2 - 2\rho R\cos\phi}\right)\bigg|_{\rho=b}$$

This is

$$\sigma = -\frac{\tau}{2\pi}\left(\frac{b - R'\cos\phi}{b^2 + R'^2 - 2bR'\cos\phi} - \frac{b - R\cos\phi}{b^2 + R^2 - 2bR\cos\phi}\right)$$

Plugging in $R' = b^2/R$ and multiplying R^2/b^2 on the first term yields:

$$\sigma = -\frac{\tau}{2\pi}\left(\frac{b - \frac{b^2}{R}\cos\phi}{b^2 + \frac{b^4}{R^2} - 2\frac{b^3}{R}\cos\phi} - \frac{b - R\cos\phi}{b^2 + R^2 - 2bR\cos\phi}\right)$$

$$\sigma = -\frac{\tau}{2\pi}\left(\frac{\frac{R^2}{b} - b}{b^2 + R^2 - 2bR\cos\phi}\right)$$

$$\sigma = \frac{\tau}{2\pi}\left(\frac{-(\frac{R}{b})^2 + 1}{b + (\frac{R}{b})^2 b - 2R\cos\phi}\right)$$

$$\sigma = \frac{\tau}{2\pi b}\left(\frac{-(\frac{R}{b})^2 + 1}{1 + (\frac{R}{b})^2 - 2(\frac{R}{b})\cos\phi}\right)$$

Where I have rearranged it for R/b and units $\tau/2\pi b$. So I have:

$$\sigma(R/b = 2) = -\frac{3}{5 - 4\cos\phi}$$

$$\sigma(R/b = 4) = -\frac{15}{17 - 8\cos\phi}$$

You can graph this on any software, or on any java applet. I used this site

http://www.shodor.org/interactivate/activities/sketcher/

d. This is the force between two wires. No need to worry about the cylinder situation, just weld the power of images and understand that the force is

$$F = qE$$

and the electric field of an infinite wire is

$$E = \frac{\tau}{2\pi\epsilon_0 s}$$

This is equation (2.9) in Griffiths. Here s is the distance from the wire, in our case we are $R - R'$ away from the image wire, so using $R' = b^2/R$:

$$E = \frac{\tau'}{2\pi\epsilon_0(R - R')} = \frac{\tau'}{2\pi\epsilon_0(R - b^2/R)} = \frac{\tau' R}{2\pi\epsilon_0(R^2 - b^2)}$$

Since we want force per unit length

$$\frac{F}{l} = \tau E$$

So because $\tau = -\tau$ we have an attractive force per unit length of:

$$\frac{F}{l} = -\frac{\tau^2 R}{2\pi\epsilon_0(R^2 - b^2)}$$