Field Theory 263: Problem Set 1

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Problem 1:

Show that Pauli's equation,

$$[\frac{1}{2m}(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{r}))^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}(\vec{r})]_{\alpha\beta}\psi_\beta(\vec{r}) = i\hbar\frac{\partial}{\partial t}\psi_\alpha(\vec{r})$$

can be more compactly written as

$$\frac{1}{2m}[(\vec{\pi}(\vec{r})\cdot\vec{\sigma})(\vec{\pi}(\vec{r})\cdot\vec{\sigma})]_{\alpha\beta}\psi_\beta(\vec{r}) = i\hbar\frac{\partial}{\partial t}\psi_\alpha(\vec{r})$$

where $\vec{\pi} = \frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{r})$.

Show that the equation in a rotated frame,($r'_i = R_{ij}r_j$ etc.)

$$\frac{1}{2m}[(\vec{\pi}'(\vec{r'})\cdot\vec{\sigma})(\vec{\pi}'(\vec{r'})\cdot\vec{\sigma})]_{\alpha\beta}\psi'_\beta(\vec{r'}) = i\hbar\frac{\partial}{\partial t}\psi'_\alpha(\vec{r'})$$

is indeed given by the rotated form of the original equation, i.e.,

$$\mathcal{R}_{\alpha\beta}(\frac{1}{2m}[(\vec{\pi}(\vec{r})\cdot\vec{\sigma})(\vec{\pi}(\vec{r})\cdot\vec{\sigma})]_{\gamma\beta}\psi_\beta(\vec{r}) - i\hbar\frac{\partial}{\partial t}\psi_\gamma(\vec{r})) = 0$$

This demonstrates the rotational covariance of the Pauli equation.
b. Show that the following Maxwell's equation

$$\partial_\nu F^{\mu\nu} = \frac{e}{c}J^\mu$$

is Lorentz covariant, i.e., the equation in another Lorentz frame

$$\partial'_\nu F'^{\mu\nu} = \frac{e}{c}J'^\mu$$

is equivalent to the Lorentz transform of the original equation

$$\Lambda^\mu_\nu(\partial_\alpha F^{\beta\alpha} - \frac{e}{c}J^\beta) = 0$$

Answer:

Using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ or also from the fundamental product rule of geometric algebra, we obtain the familiar relation,

$$(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B)$$

With

$$\frac{1}{2m}[(\vec{\pi}(\vec{r}) \cdot \vec{\sigma})(\vec{\pi}(\vec{r}) \cdot \vec{\sigma})]_{\alpha\beta}\psi_\beta(\vec{r}) = i\hbar\frac{\partial}{\partial t}\psi_\alpha(\vec{r})$$

we have

$$\frac{1}{2m}[\pi^2 + i\sigma \cdot (\pi \times \pi)]_{\alpha\beta}\psi_\beta = i\hbar\partial_t\psi_\alpha$$

where the vector arrows and their respective functions have been supressed. Plugging in for π

$$\frac{1}{2m}[(p - \frac{e}{c}A)^2 + i\sigma \cdot (p - \frac{e}{c}A) \times (p - \frac{e}{c}A)]_{\alpha\beta}\psi_\beta = i\hbar\partial_t\psi_\alpha$$

$$\frac{1}{2m}[(p - \frac{e}{c}A)^2 - i\frac{e}{c}\sigma \cdot (p \times A + A \times p)]_{\alpha\beta}\psi_\beta = i\hbar\partial_t\psi_\alpha$$

The cross product terms can be evaluated, as is also done in Ryder p.54 by,

$$[p_i, A_j] = -i\hbar\partial_i A_j$$

Subtracting,

$$[p_i, A_j] - [p_j, A_i] = -i\hbar(\partial_i A_j - \partial_j A_i)$$

Multiplying by ϵ_{ijk},

$$(p_i A_j - p_j A_i)\epsilon_{ijk} + (A_i p_j - A_j p_i)\epsilon_{ijk} = -i\hbar(\partial_i A_j - \partial_j A_i)\epsilon_{ijk}$$

Summing over i and j give the k component of $-i\hbar\vec{B}$, therefore, we can use

$$p \times A + A \times p = -i\hbar\nabla \times A = -i\hbar B$$

I now have

$$\frac{1}{2m}[(p - \frac{e}{c}A)^2 - i\frac{e}{c}\sigma \cdot (-i\hbar B)]_{\alpha\beta}\psi_\beta = i\hbar\partial_t\psi_\alpha$$

which, unsupressing vector signs and functional dependence, is

$$[\frac{1}{2m}(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{r}))^2 - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B}(\vec{r})]_{\alpha\beta}\psi_\beta(\vec{r}) = i\hbar\frac{\partial}{\partial t}\psi_\alpha(\vec{r})$$

To demonstrate the rotational covariance of the Pauli equation, we need to show that this:

$$\mathcal{R}_{\alpha\beta}(\frac{1}{2m}[(\vec{\pi}(\vec{r})\cdot\vec{\sigma})(\vec{\pi}(\vec{r})\cdot\vec{\sigma})]_{\gamma\beta}\psi_\beta(\vec{r}) - i\hbar\frac{\partial}{\partial t}\psi_\gamma(\vec{r})) = 0$$

equals this:

$$\frac{1}{2m}[(\vec{\pi'}(\vec{r'})\cdot\vec{\sigma})(\vec{\pi'}(\vec{r'})\cdot\vec{\sigma})]_{\alpha\beta}\psi'_\beta(\vec{r'}) = i\hbar\frac{\partial}{\partial t}\psi'_\alpha(\vec{r'})$$

In order to simplify my life, I'm going to drop the functional dependence notation, the vector hats, and for a trial run, even the indices. This will better demonstrate the overall idea. Watch out for the indicies, as they can make life harder than it really is. The solution sketch goes like this:

$$\mathcal{R}\{[\pi\cdot\sigma)(\pi\cdot\sigma)]\psi\} = \frac{\partial}{\partial t}\psi'$$

$$\mathcal{R}\{[\pi\cdot\sigma)(\pi\cdot\sigma)]\mathcal{R}^\dagger\psi'\} = \frac{\partial}{\partial t}\psi'$$

$$\mathcal{R}\{[\pi\cdot\sigma)\mathcal{R}^\dagger\mathcal{R}(\pi\cdot\sigma)]\mathcal{R}^\dagger\psi'\} = \frac{\partial}{\partial t}\psi'$$

$$(\pi\cdot\sigma R)(\pi\cdot\sigma R)\psi' = \frac{\partial}{\partial t}\psi'$$

$$(\pi'\cdot\sigma)(\pi'\cdot\sigma)\psi' = \frac{\partial}{\partial t}\psi'$$

Now, in order to make things grossly explicit, I'm going to put in indicies and keep track in glory detail. First I need to change ψ_β

$$\psi_\beta = \delta_{\beta\lambda}\psi_\lambda = \mathcal{R}^\dagger_{\beta\rho}\mathcal{R}_{\rho\lambda}\psi_\lambda = \mathcal{R}^\dagger_{\beta\rho}\psi'_\rho$$

So we have, including the indicies and simply rotating the right side term,

$$\mathcal{R}_{\alpha\gamma}\{\frac{1}{2m}[(\pi_i\sigma_i)(\pi_k\sigma_k)]_{\gamma\beta}\psi_\beta\} = i\hbar\frac{\partial}{\partial t}\psi'_\alpha$$

Substituting my ψ_β, the left hand term becomes

$$\mathcal{R}_{\alpha\gamma}\frac{1}{2m}[(\pi_i\sigma_i)(\pi_k\sigma_k)]_{\gamma\beta}\mathcal{R}^\dagger_{\beta\rho}\psi'_\rho$$

Inserting an identity matrix right smack in the middle, we have

$$\frac{1}{2m}\mathcal{R}_{\alpha\gamma}(\pi_i\sigma_i)_{\gamma\delta}\mathcal{R}^\dagger_{\delta\lambda}\mathcal{R}_{\lambda\epsilon}(\pi_k\sigma_k)_{\epsilon\beta}\mathcal{R}^\dagger_{\beta\rho}\psi'_\rho = i\hbar\frac{\partial}{\partial t}\psi'_\alpha$$

If I use the property, $\sigma_i R_{ik} = \mathcal{R}\sigma_k\mathcal{R}^\dagger$, I conclude that the left hand side becomes

$$\frac{1}{2m}(\pi_i\sigma_j R_{ji})_{\alpha\lambda}(\pi_k\sigma_i R_{ik})_{\lambda\rho}\psi'_\rho$$

Using, $\pi'_j = R_{ji}\pi_i$, this becomes

$$\frac{1}{2m}(\pi'_j\sigma_j)_{\alpha\lambda}(\pi'_i\sigma_i)_{\lambda\rho}\psi'_\rho$$

Lets simplify those indicies and our equation is now

$$\frac{1}{2m}[(\pi'_j\sigma_j)(\pi'_i\sigma_i)]_{\alpha\rho}\psi'_\rho = i\hbar\frac{\partial}{\partial t}\psi'_\alpha$$

As the ρ's are of course just dummies, we have

$$\frac{1}{2m}[(\vec{\pi'}(\vec{r'})\cdot\vec{\sigma})(\vec{\pi'}(\vec{r'})\cdot\vec{\sigma})]_{\alpha\beta}\psi'_\beta(\vec{r'}) = i\hbar\frac{\partial}{\partial t}\psi'_\alpha(\vec{r'})$$

which demonstrates the rotational covariance of the Pauli equation.

Part b.

If we evaluate

$$\Lambda^\mu{}_\beta(\partial_\alpha F^{\beta\alpha} - \frac{e}{c}J^\beta) = 0$$

we have

$$\Lambda^\mu{}_\beta\partial_\alpha F^{\beta\alpha} - \Lambda^\mu{}_\beta\frac{e}{c}J^\beta = 0$$

$$\Lambda^\mu{}_\beta\partial_\alpha F^{\beta\alpha} - \frac{e}{c}J'^\mu = 0$$

$$\Lambda^\mu{}_\beta\partial_\alpha F^{\beta\alpha} = \frac{e}{c}J'^\mu$$

Using, $\Lambda^\nu{}_\alpha\partial^\alpha = \partial'^\nu$, and applying an inverse transformation, yields,

$$\Lambda_\nu{}^\alpha\partial'^\nu = \Lambda_\nu{}^\alpha\Lambda^\nu{}_\alpha\partial^\alpha$$

$$\Lambda_\nu{}^\alpha\partial'^\nu = \partial^\alpha$$

$$\Lambda^\nu{}_\alpha\partial'_\nu = \partial_\alpha$$

Plugging this into ∂_α,

$$\Lambda^\mu{}_\beta(\Lambda^\nu{}_\alpha\partial'_\nu)F^{\beta\alpha} = \frac{e}{c}J'^\mu$$

$$\partial'_\nu\Lambda^\mu{}_\beta\Lambda^\nu{}_\alpha F^{\beta\alpha} = \frac{e}{c}J'^\mu$$

$$\partial'_\nu F'^{\mu\nu} = \frac{e}{c}J'^\mu$$

therefore this Maxwell equation is Lorentz covariant.

Problem 2:
The Klein Paradox. Consider a one dimensional potential barriar problem for the KG equation for which

$$H_{classical} = \sqrt{p^2 + m^2} + V$$

$$V = \begin{cases} V_0 & x > 0 \\ 0 & x < 0 \end{cases}$$

For stationary state solutions

$$\phi(x,t) = e^{-iEt} f(x)$$

we have

$$f(x) = Ae^{ikx} + Be^{-ikx} \qquad k = \sqrt{E^2 - m^2} \quad x < 0$$
$$f(x) = Ce^{iKx} \qquad K = \sqrt{(E - V_0)^2 - m^2} \quad x > 0$$

We expect that for $V_0 > E - m = \textit{kinetic energy}$ (the non-classically allowed case) K is imaginary so that $f(x)$ is purely damped. Show that this expectation is NOT borne out for $V_0 > E + m$. Draw the energy spectrum for both regions $x < 0$ and $x > 0$; try to explain the paradox.

Answer:

The expectation that K is imaginary is not borne out when $V_0 > E + m$ because K^2 is now positive. This can be seen by just plugging

$$V_0 > E + m$$

into

$$K^2 = (E - V_0)^2 - m^2$$

where the first term will now be larger than m^2. We would normally expect the wavefunction to be even more strongly damped, since $V_0 > E + m$, but because $K^2 > 0$ we have a region which is oscillatory like the region $x < 0$. Basically, a particle can be confined in the region $x < 0$ and tunnel through the region where $x > 0$ and $V_0 > E - m$, and then behave like it was in an attractive potential in region $x > 0$ and $V > E + m$. Despite the fact that $V > E + m$ seemingly implies it would behave like a particle in a very strong repulsive potential.

The free-particle solutions to the Dirac equation exhibit an energy spectrum from $-m$ to $-\infty$ and from m to ∞. The condition to have negative energy oscillatory solutions with an applied small positive potential is:

$$-\infty < E < -m + V$$

From this we can see that E doesn't have to be negative. One way to explain the Klein-Gordon paradox is that we have a negative energy solution despite $E > 0$. We would just increase V enough so an oscillatory negative energy solution can have the same positive energy as an oscillatory positive energy solution where $x < 0$.

Problem 4: a. Show that $\partial_{c\dot{c}}\epsilon^{\dot{c}\dot{a}}\partial_{a\dot{a}}\epsilon^{ab}\phi_b = \partial^2\phi_c$. b. Show that if ϕ obeys $(-\partial^2 + m^2)\phi_a = 0$, so does $\dot{\chi}$.

Answer:

Substituting $\partial_{a\dot{b}} = (\sigma_\mu)_{a\dot{b}}\partial^\mu$ and $\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = i\sigma_2$ we will have

$$\begin{aligned}
\partial_{c\dot{c}}\epsilon^{\dot{c}\dot{a}}\partial_{a\dot{a}}\epsilon^{ab}\phi_b &= (\sigma_\mu)_{c\dot{c}}\partial^\mu(i\sigma_2)^{\dot{c}\dot{a}}(\sigma_\nu)_{a\dot{a}}\partial^\nu(i\sigma_2)^{ab}\phi_b \\
&= (\sigma_\mu)_{c\dot{c}}(i\sigma_2)^{\dot{c}\dot{a}}(\sigma_\nu)_{a\dot{a}}(i\sigma_2)^{ab}\partial^\mu\partial^\nu\phi_b \\
&= (\sigma_\mu)_{c\dot{c}}(i\sigma_2)^{\dot{c}\dot{a}}(\sigma_\nu^T)_{\dot{a}a}(i\sigma_2)^{ab}\partial^\mu\partial^\nu\phi_b \\
&= (\sigma_\mu)_{c\dot{c}}(\sigma_\nu)^{\dot{c}b}\partial^\mu\partial^\nu\phi_b
\end{aligned}$$

Where I used $\sigma_2\sigma_i^T\sigma_2 = -\sigma_i$ in the last step. Now applying the anticommunitive property of the matrices,

$$\{\sigma_\mu, \sigma_\nu\} = 2\delta^\mu_\nu$$

Then I've got

$$(\sigma_\mu\sigma_\nu)^b_c = \frac{1}{2}\{\sigma_\mu, \sigma_\nu\}^b_c = \frac{1}{2}\cdot 2\{\delta_{\mu\nu}\}^b_c = \{\delta_{\mu\nu}\}^b_c$$

Therefore

$$\begin{aligned}(\sigma_\mu)_{c\dot{c}}(\sigma_\nu)^{\dot{c}b}\partial^\mu\partial^\nu\phi_b &= (\delta_{\mu\nu})^b_c\partial^\mu\partial^\nu\phi_b \\ &= \partial^\mu\partial_\mu\phi_c \\ &= \partial^2\phi_c\end{aligned}$$

which is what we set out to prove. $\boxed{\partial_{c\dot{c}}\epsilon^{\dot{c}\dot{a}}\partial_{a\dot{a}}\epsilon^{ab}\phi_b = \partial^2\phi_c}$.

For part b, we know that ϕ satisfies $(-\partial^2 + m^2)\phi_a = 0$, so to prove

$$(-\partial^2 + m^2)\dot{\chi}_c = 0$$

we shall start with

$$\partial_{b\dot{b}}\epsilon^{\dot{b}\dot{a}}\dot{\chi}_{\dot{a}} = m\phi_b$$

and multiply by $\partial_{c\dot{c}}\epsilon^{cb}$ to get

$$\partial_{c\dot{c}}\epsilon^{cb}\partial_{b\dot{b}}\epsilon^{\dot{b}\dot{a}}\dot{\chi}_{\dot{a}} = \partial_{c\dot{c}}\epsilon^{cb}m\phi_b$$

Applying our property that we proved in part (a), we may write the left hand side as

$$\partial_{c\dot{c}}\epsilon^{cb}\partial_{b\dot{b}}\epsilon^{\dot{b}\dot{a}}\dot{\chi}_{\dot{a}} = \partial^2\dot{\chi}_{\dot{c}}$$

So we have

$$\partial^2\dot{\chi}_{\dot{c}} = \partial_{c\dot{c}}\epsilon^{cb}m\phi_b$$

Using $\partial_{c\dot{c}}\epsilon^{cb}\phi_b = m\dot{\chi}_{\dot{c}}$,

$$\partial^2\dot{\chi}_{\dot{c}} = m^2\dot{\chi}_{\dot{c}}$$

This is

$$\boxed{(-\partial^2 + m^2)\dot{\chi}_{\dot{c}} = 0}$$

Problem 5:

a. Denote the set of 16 γ-matrices by $P = (1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu})$. Find their trace. Show that $(1, \gamma_5, \gamma_\mu)$ are linearly independent.

b. Simplify $\gamma^\mu\gamma_\mu$, $\gamma^\mu\gamma^\nu\gamma_\mu$, and $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu$.

c. Find $Tr(\gamma^\mu\gamma^\nu)$, $Tr(\gamma^\gamma\gamma^\nu\gamma^l)$, and $Tr(\gamma^\mu\gamma^\nu\gamma^l\gamma^\sigma)$.

Answer:

$$\boxed{Tr(\mathbf{1}) = 4}$$

as these are four by four matrices. In a particular representation, say the standard Dirac-Pauli representation, we have for γ_5:

$$Tr(\gamma_5) = Tr\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} = 0$$

but this is true as the trace of any odd number of gamma matrices is zero.

$$Tr(\gamma_5) = Tr(\gamma_1\gamma_1\gamma_5) = -Tr(\gamma_1\gamma_5\gamma_1) = -Tr(\gamma^1\gamma^1\gamma_5) = -Tr(\gamma_5)$$

$$\boxed{Tr(\gamma_5) = 0}$$

$$Tr(\gamma_\mu) = Tr(\gamma_5\gamma_5\gamma_\mu) = -Tr(\gamma_5\gamma_\mu\gamma_5) = -Tr(\gamma_5\gamma_5\gamma_\mu) = -Tr(\gamma_\mu)$$

As it is equal to its negative, it must vanish.

$$\boxed{Tr(\gamma_\mu) = 0}$$

$$Tr(\gamma_5\gamma_\mu) = -Tr(\gamma_\mu\gamma_5) = -Tr(\gamma_5\gamma_\mu)$$

Via anticommunitivity, for the first step and cyclic property of the trace for the second step.

$$\boxed{Tr(\gamma_5\gamma_\mu) = 0}$$

The four by four spin matrices, contain

$$Tr(\sigma_{\mu\nu}) = Tr(\frac{i}{2}[\gamma_\mu, \gamma_\nu]) = \frac{i}{2}Tr(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

Now for any $n \times n$ matrices U and V

$$Tr(UV) = Tr(VU)$$

Thus

$$\boxed{Tr(\sigma_{\mu\nu}) = 0}$$

Linear independence of $\mathbf{1}$, γ_5 and γ_μ means if

$$A\mathbf{1} + B\gamma_5 + C\gamma_m u = 0$$

then

$$A = B = C = 0$$

I can show $A = 0$ by multiplying by $\mathbf{1}$ and taking the trace:

$$Tr(A\mathbf{1} + B\gamma_5 + C\gamma_\mu) = 0$$

$$Tr(A\mathbf{1}) + Tr(B\gamma_5) + Tr(C\gamma_\mu) = 0$$

and from above, the last two terms where shown to be zero,

$$Tr(A\mathbf{1}) = 0$$

$$\boxed{A = 0}$$

If $A = 0$ then $B\gamma_5 + C\gamma_\mu = 0$ must imply B and C are zero for linear independence. Multiplying by $\mathbf{1}\gamma_5$ and taking the trace again,

$$B\mathbf{1} + C\gamma_\mu\gamma_5 = 0$$

$$Tr(B\mathbf{1}) = -Tr(C\gamma_\mu\gamma_5)$$

$$Tr(B\mathbf{1}) = 0$$

$$\boxed{B = 0}$$

Thus we have left,

$$C\gamma_\mu = 0$$

which implies

$$\boxed{C = 0}$$

And we have shown that $\mathbf{1}$, γ_5 and γ_μ are linaer independent.
b. Here we have

$$\gamma^\mu \gamma_\mu = \eta_{\mu\nu}\gamma^\mu \gamma^\nu = \frac{1}{2}\eta_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \eta_{\mu\nu}\eta^{\mu\nu} = 4$$

where I used $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.
So we've got:

$$\boxed{\gamma^\mu \gamma_\mu = 4}$$

It's not hard to simplify $\gamma^\mu \gamma^\nu \gamma_\mu$, just apply the anticommutivity property,

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2\eta^{\mu\nu} - \gamma^\nu \gamma^\mu)\gamma_\mu = 2\gamma^\nu - \gamma^\nu \gamma^\mu \gamma_\mu$$

and from the above box,

$$\gamma^\mu \gamma^\nu \gamma_\mu = 2\gamma^\nu - \gamma^\nu \cdot 4$$

$$\boxed{\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu}$$

The same process goes for $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu$:

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = (2\eta^{\mu\nu} - \gamma^\nu \gamma^\mu)(\gamma^\rho \gamma_\mu)$$

Using the above box

$$2\gamma^\rho \gamma^\nu - \gamma^\nu \gamma^\mu \gamma^\rho \gamma_\mu = 2\gamma^\rho \gamma^\nu - \gamma^\nu(-2\gamma^\rho) = 2(\gamma^\rho \gamma^\nu + \gamma^\nu \gamma^\rho)$$

applying the anticommuntivity property,

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 2\{\gamma^\rho, \gamma^\nu\} = 4\eta^{\rho\nu}$$

$$\boxed{\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho}}$$

c. The trace can be found though anticommutation and cyclicity.

$$Tr(\gamma^\mu \gamma^\nu) = Tr(2\eta^{\mu\nu} \cdot \mathbf{1} - \gamma^\nu \gamma^\mu)$$

Cycle the last term and recall the $Tr(\mathbf{1}) = 4$:

$$Tr(\gamma^\mu \gamma^\nu) = 2\eta^{\mu\nu} \cdot 4 - Tr(\gamma^\mu \gamma^\nu)$$

Bring the last term to the other side and viola:

$$\boxed{Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}}$$

Next, any odd number gamma trace will be zero, for this three gamma trace we have,

$$Tr(\gamma^\gamma \gamma^\nu \gamma^\rho) = Tr(\gamma^5 \gamma^5 \gamma^\gamma \gamma^\nu \gamma^\rho)$$

as $\gamma^5\gamma^5 = 1$, and since $\{\gamma^5, \gamma^\mu\} = 0$ I can say

$$Tr(\gamma^\gamma\gamma^\nu\gamma^\rho) = -Tr(\gamma^5\gamma^\gamma\gamma^\nu\gamma^\rho\gamma^5)$$

Cycle through

$$Tr(\gamma^\gamma\gamma^\nu\gamma^\rho) = -Tr(\gamma^5\gamma^5\gamma^\gamma\gamma^\nu\gamma^\rho) = -Tr(\gamma^\gamma\gamma^\nu\gamma^\rho)$$

As the trace equals the negative of itself, it must vanish:

$$\boxed{Tr(\gamma^\gamma\gamma^\nu\gamma^\rho) = 0}$$

Lastly, we can evaluate $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ through repeated use of the anticommunitivity relation.

$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = Tr(2\eta^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma)$$

Apply it again,

$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = Tr(2\eta^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2\eta^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma)$$

and again,

$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = Tr(2\eta^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2\eta^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho 2\eta^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)$$

Bring the last term over to the otherside after cycling it once and we get

$$2 \cdot Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2 \cdot Tr(\eta^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\eta^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho\eta^{\mu\sigma})$$

Using our $Tr(\gamma^\mu\gamma^\nu) = 2\eta^{\mu\nu}$ from above,

$$\boxed{Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})}$$