# Homework 1: # 4.1, 4.2, 4.3, 4.4, 4.8, 4.10, 4.12

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Problem 4.1

Calculate the three lowest energy levels, together with their degeneracies, for the following systems (assume equal mass distinguishable particles):
a. Three noninteracting spin $\frac{1}{2}$ particles in a box of length $L$.
b. Four noninteracting spin $\frac{1}{2}$ particles in a box of length $L$.

Solution:

In a box we have three dimensions. So the energy for one particle is

$$E^1_{n_x,n_y,n_z} = (n_x^2 + n_y^2 + n_z^2)\frac{\pi^2\hbar^2}{2mL^2}$$

If the particles are distinguishable, the composite wave functions are products, as explained in an example by Griffiths, section 5.1. Therefore the energy of three particles are

$$E_T = E^1 + E^2 + E^3$$

This is

$$E_T = \sum_{i,j}^{3} n_{ij}^2 \frac{\pi^2\hbar^2}{2mL^2}$$

where the index $i$ represents the particle, and $j$ represents the dimension. Note that there are 9 distinct $n$'s. The $n$'s are non-zero integers, so the lowest energy is where all the $n$ values are equal to 1. That is just $1^2 * 9 = 9$

$$E_1 = \frac{9\pi^2\hbar^2}{2mL^2}$$

The second lowest energy is where only one of the $n$ values is 2 and the rest are still 1. That is $2^2 + 8 = 12$. So

$$E_2 = \frac{12\pi^2\hbar^2}{2mL^2}$$

The third lowest energy is where two of the $n$'s have value 2 and rest are still 1. That is $2^2 + 2^2 + 7 = 15$. So

$$E_3 = \frac{15\pi^2\hbar^2}{2mL^2}$$

The degeneracy is found by multiplying the number of spatial functions by the number of spin functions. There are 8 different spin functions.

$$|---\rangle, |--+\rangle, |-+-\rangle, |+--\rangle, |++-\rangle, |-++\rangle, |+-+\rangle, |+++\rangle$$

or $2^3 = 8$ combinations. The spatial functions are found by looking at the possibilities for the $n$'s. For $E_1$ there is only one possibility, that is, with all the $n$ values equal to 1. For $E_2$ there are 9 possibilities, where each of the 9 values of $n$ could be 2 where the rest are 1. For $E_3$ its a little more difficult, because two of the $n$'s may have the value 2 so we use the combination formula '9 choose 2'.

$$\frac{N!}{(N-n)!n!} = \frac{9!}{(9-2)!2!} = 36$$

possibilities for $E_3$. Multiplying the spatial possibilities times the spin possibilities gives the degeneracy.

$$E_1 \rightarrow 1 * 8 = 8$$
$$E_2 \rightarrow 9 * 8 = 72$$
$$E_3 \rightarrow 36 * 8 = 288$$

For four particles we have $4 * 3 = 12$ distinct $n$'s. So for the lowest energy, all of them are 1:

$$E_1 = \frac{12\pi^2\hbar^2}{2mL^2}$$

For the second lowest energy we have $2^2 + 11 = 15$:

$$E_2 = \frac{15\pi^2\hbar^2}{2mL^2}$$

For the third lowest energy we have $2^2 + 2^2 + 10 = 18$:

$$E_3 = \frac{18\pi^2\hbar^2}{2mL^2}$$

There are $2^4 = 16$ spin possibilities. There is only 1 spatial possibility for $E_1$. There are 12 spatial possibilities for $E_2$. And '12 choose 2' possibilities for $E_3$

$$\frac{12!}{(12-2)!2!} = 66$$

The degeneracies are therefore

$$E_1 \to 1 * 16 = 16$$

$$E_2 \to 12 * 16 = 192$$

$$E_3 \to 66 * 16 = 1056$$

Problem 4.2

Let $\mathcal{T}_d$ denote the translation operator (displacement vector $\vec{d}$); $\mathcal{D}(\hat{n}, \phi)$, the rotation operator ($\hat{n}$ and $\phi$ are the axis and angle of rotation, respectively); and $\pi$ the parity operator. Which, if any, of the following pairs commute? Why?

- $\mathcal{T}_d$ and $\mathcal{T}_{d'}$ ($\vec{d}$ and $\vec{d'}$ in different directions).
- $\mathcal{D}(\hat{n}, \phi)$ and $\mathcal{D}(\hat{n}', \phi')$ ($\hat{n}$ and $\hat{n}'$ in different directions).
- $\mathcal{T}_d$ and $\pi$.
- $\mathcal{D}(\hat{n}, \phi)$ and $\pi$.

Solution:

a. $\mathcal{T}_d$ and $\mathcal{T}_{d'}$ commute as can be seen graphically by translating a vector in two directions in different orders or algebraically(where $r$ and $d$ are vectors) by

$$\mathcal{T}_d \phi(r) = \phi(r+d) \to \mathcal{T}_{d'} \mathcal{T}_d \phi(r) = \phi(r+d+d')$$

$$\mathcal{T}_{d'} \phi(r) = \phi(r+d') \to \mathcal{T}_d \mathcal{T}_{d'} \phi(r) = \phi(r+d'+d)$$

Therefore

$$[\mathcal{T}_d, \mathcal{T}_{d'}]\phi(r) = 0$$

and

$$[\mathcal{T}_d, \mathcal{T}_{d'}] = 0$$

because $\phi(r)$ can be any function of vector $r$.

b. $\mathcal{D}(\hat{n}, \phi)$ and $\mathcal{D}(\hat{n}', \phi')$ do not commute. As we know that rotations along different axis do not commute and $\mathcal{D}$ acts on $\phi$ to cause a rotation. This can be done by rotating a book around two different axis.

c. $\mathcal{T}_d$ and $\pi$ do not commute. This can be done graphically by translating a vector, then mirroring it, and see if that point matches up if you mirror the vector then translate it. Algebraically,

$$\pi\phi(r) = \phi(-r) \rightarrow \mathcal{T}_d\pi\phi(r) = \phi(-r+d)$$

$$\mathcal{T}_d\phi(r) = \phi(r+d) \rightarrow \pi\mathcal{T}_d\phi(r) = \phi(-r-d)$$

Therefore

$$[\mathcal{T}_d, \pi]\phi(r) \neq 0$$

$$[\mathcal{T}_d, \pi] \neq 0$$

d. $\mathcal{D}(\hat{n}, \phi)$ and $\pi$ commute. Graphically by rotating then mirroring, and checking to see if that matches up with mirroring then rotating. Algebraically,

$$\mathcal{D}\phi(r) = \phi(x) \rightarrow \pi\mathcal{D}\phi(r) = \phi(-x)$$

$$\pi\phi(r) = \phi(-r) \rightarrow \mathcal{D}\pi\phi(r) = \mathcal{D}\phi(-r) = \phi(-x)$$

Therefore

$$[\mathcal{D}, \pi]\phi(r) = 0$$

$$[\mathcal{D}, \pi] = 0$$

Problem 4.3

A quantum-mechanical state $\Psi$ is known to be simultaneous eigen-state of two Hermitian operators $A$ and $B$ which anticommute,

$$AB + BA = 0$$

What can you say about the eigenvalues of $A$ and $B$ for stat $\Psi$? Illustrate your point using the parity operator (which can be chosen to satisfy $\pi = \pi^{-1} = \pi^{\dagger}$) and the momentum operator.

Solution:

Taking a look at the eigenvalues we see

$$A\psi = a\psi \quad B\psi = b\psi$$

$$B(A\psi) = B(a\psi) = ab\psi$$
$$A(B\psi) = A(b\psi) = ba\psi$$

And we have

$$AB\psi = -BA\psi$$

So

$$ba\psi = -ab\psi \qquad (ba + ab)\psi = 0$$

and since $\psi$ is an arbitrary test function, we may say

$$ba + ab = 0$$

Now $b$ and $a$ are just numbers, the eigenvalues of $B$ and $A$. We thus have

$$ab + ab = 0 \rightarrow 2ab = 0$$

and

$$ab = 0$$

This is only the case if $a = 0$ or $b = 0$. If I assign $A \rightarrow \pi$ and $B \rightarrow P$, analogously I have

$$\pi P + P\pi = \{\pi, P\} = 0$$

If the eigenvalue of the momentum operator is $p = 0$, or the eigenvalue of the parity operator is zero, then the momentum eigenstate is a parity eigenstate. This probably does not happen unless one of the eigenvalues is zero.

Problem 4.4

A spin $\frac{1}{2}$ particle is bound to a fixed center by a spherically symmetrical potential.
a. Write down the spin angular function $\mathcal{Y}_{l=0}^{j=1/2,m=1/2}$.
b. Express $(\boldsymbol{\sigma} \cdot \boldsymbol{x})\mathcal{Y}_{l=0}^{j=1/2,m=1/2}$ in terms of some other $\mathcal{Y}_{l}^{j,m}$.
c. Show that your result in (b) is understandable in view of the transformation properties of the operator $\boldsymbol{S} \cdot \boldsymbol{x}$ under rotations and under space inversion(parity).

Solution:

a. The particle is spin $\frac{1}{2}$. Therefore $j = s + l = \frac{1}{2} + 0 = +\frac{1}{2}$. Using equation 3.7.64 in Sakurai, I'll plug in $l = 0$,$j = +\frac{1}{2}$ and $m = \frac{1}{2}$.

$$\mathcal{Y}_{l=0}^{j=1/2,m=1/2} = \begin{pmatrix} \mathcal{Y}_0^0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

b. Using the Pauli spin matrices and the vector $\boldsymbol{x} = x\hat{i} + y\hat{j} + z\hat{k}$ I convert

$$\boldsymbol{\sigma} \cdot \boldsymbol{x} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

So we have

$$(\boldsymbol{\sigma} \cdot \boldsymbol{x})\mathcal{Y}_{l=0}^{j=1/2,m=1/2} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} z \\ x + iy \end{pmatrix}$$

Now a good idea would be to recognize that because the spin angular momentum functions are expressed in terms of spherical coordinates, we should convert this matrix to spherical coordinates using

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

So

$$\begin{pmatrix} z \\ x+iy \end{pmatrix} = r\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}$$

Using 3.6.34 and 3.6.39 from Sakurai we've got

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

So I'll plug and chug

$$(\boldsymbol{\sigma}\cdot\boldsymbol{x})\mathcal{Y}_{l=0}^{j=1/2,m=1/2} = \frac{r}{\sqrt{4\pi}}\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} = \frac{-r}{\sqrt{4\pi}}\begin{pmatrix} -\sqrt{\frac{4\pi}{3}}Y_1^0 \\ \sqrt{\frac{8\pi}{3}}Y_1^1 \end{pmatrix} = \frac{-r}{\sqrt{3}}\begin{pmatrix} -Y_1^0 \\ \sqrt{2}Y_1^1 \end{pmatrix}$$

To express this in terms of some $\mathcal{Y}$ I'll use equation 3.7.64 again and note that only if I use $l=1$, $m=1/2$, and the lower sign, will I get the expression above. Thus

$$\frac{-r}{\sqrt{3}}\begin{pmatrix} -Y_1^0 \\ \sqrt{2}Y_1^1 \end{pmatrix} = -r\mathcal{Y}_{l=1}^{j=1/2,m=1/2}$$

and

$$(\boldsymbol{\sigma}\cdot\boldsymbol{x})\mathcal{Y}_{l=0}^{j=1/2,m=1/2} = -r\mathcal{Y}_{l=1}^{j=1/2,m=1/2}$$

c. This result makes sense in view of the properties of $\boldsymbol{S}\cdot\boldsymbol{x}$ because it's a pseudoscalar. It acts like a scalar under rotation and scalar operators cannot change $j,m$ values shown in Example 1, pg240. Our $j,m$ values did not change. But under space inversions

$$\pi^{-1}\boldsymbol{S}\cdot\boldsymbol{x}\pi = -\boldsymbol{S}\cdot\boldsymbol{x}$$

This shows that $\boldsymbol{S}\cdot\boldsymbol{x}$ is odd. The $l=0$ function is even, and the $l=1$ function is odd. So it would seem to make sense that $\boldsymbol{S}\cdot\boldsymbol{x}$ will change the $l$ value and leave the $j,m$ values the same.

Problem 4.8

a. Assuming that the Hamiltonian is invariant under time reversal, prove that the wave function for a spinless nondegenerate system at any given instant of time can always be chosen to be real.
b. The wave function for a plane-wave state at $t = 0$ is given by a complex function $e^{\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}$. Why does this not violate time-reversal invariance?

Solution:

a. The proof is spelled out on page 277 Sakurai. For a spinless system, the wavefunction for time-reversed state is obtained by complex conjugation. This is by

$$H\Theta\psi = \Theta H\psi = E_n\Theta\psi$$

So

$$H\Theta\psi = E_n\Theta\psi$$

with

$$H\psi = E\psi$$

means $\psi$ and $\Theta\psi$ represent the same state because of the nondegeneracy assumption.(otherwise 2 different states will exist with the same energy $E_n$). Because $\psi = \langle\boldsymbol{x}'|\psi\rangle$ and $\Theta\psi = \langle\boldsymbol{x}'|\psi\rangle^*$.

$$\langle\boldsymbol{x}'|\psi\rangle = \langle\boldsymbol{x}'|\psi\rangle^*$$

So the functions are chosen to be real. As a side note we have a particular form for the time reversal, namely $\Theta = K$. For expansion in momentum $\Theta$ changes $|\boldsymbol{p}'\rangle$ into $|-\boldsymbol{p}'\rangle$:

$$\Theta\psi = \int d^3p'|-\boldsymbol{p}'\rangle\langle\boldsymbol{p}'|\psi\rangle^* = \int d^3p'|\boldsymbol{p}'\rangle\langle-\boldsymbol{p}'|\psi\rangle^*$$

Therefore

$$\psi(\boldsymbol{p}') = \langle\boldsymbol{p}'|\psi\rangle \qquad \Theta\psi = \langle-\boldsymbol{p}'|\psi\rangle^*$$

$$\psi(\boldsymbol{p}') = \psi^*(-\boldsymbol{p}')$$

Therefore $\Theta \neq K$. So, as Sakurai reminds us, the particular form of the time reversal operator depends on the particular representation used. But the wave functions may always be taken to be real, as long as we have a spinless nondegenerate system.

b. The wave function of a plane wave,

$$\psi(t = 0) = Ae^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}$$

is complex, but is does not violate the time reversal invariance because it is degenerate with: (Sakurai mentions this on pg 277):

$$\Theta\psi(t=0) = Ae^{-i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}$$

Problem 4.10

a. What is the time-reversed state corresponding to $\mathcal{D}(R)|j,m\rangle$?
b.Using the properties of time reversal and rotations, prove

$$\mathcal{D}^{(j)*}_{m'm}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R)$$

c. Prove $\Theta|j,m\rangle = i^{2m}|j,-m\rangle$.

Solution:

a. So we are looking for $\Theta\mathcal{D}(R)|j,m\rangle$. We can say generally that

$$\mathcal{D}(R) = e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi/\hbar}$$

If I insert $\Theta^{-1}\Theta = 1$, then I'll have

$$\Theta e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi/\hbar}|j,m\rangle = \Theta e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi/\hbar}\Theta^{-1}\Theta|j,m\rangle$$

Using equation 4.4.53

$$\Theta\boldsymbol{J}\Theta^{-1} = -\boldsymbol{J}$$

and the fact that the antilinearity of $\Theta$ implies $i \to -i$, we get

$$\Theta e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi/\hbar}|j,m\rangle = e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi/\hbar}\Theta|j,m\rangle =$$

Using equation 4.4.78

$$\Theta|j,m\rangle = (-1)^m|j,-m\rangle$$

we have

$$\Theta\mathcal{D}(R)|j,m\rangle = \mathcal{D}(R)(-1)^m|j,-m\rangle$$

b. The best way to do this is to look at the matrix elements. We want to prove:

$$\mathcal{D}^{(j)*}_{m'm}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R)$$

and we have from part (a), the time reversed state. So we may be tempted to say:

$$\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle = \langle j,-m'|(-1)^m\mathcal{D}(R)|j,-m\rangle$$
$$\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle = (-1)^m\mathcal{D}^{(j)}_{-m',-m}(R)$$

Almost there, we have the double negative $m$'s, we need to find what else $\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle$ is equal to. Preferably something with a complex conjugation. At the point, when you don't know what else to do, you insert a 1 into the equation and see if something comes out. Here's a good 1:

$$\sum_{j,m''}|j,m''\rangle\langle j,m''|=1$$

So our matrix looks now like this

$$\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle=\sum_{j,m''}\langle j,-m'|\Theta|j,m''\rangle\langle j,m''|\mathcal{D}(R)|j,m\rangle^*$$

The complex conjugation comes from $\Theta$ because its an antilinear operator that changes $i\to-i$. Using

$$\Theta|j,m''\rangle=(-1)^{m''}|j,-m''\rangle$$

we have

$$\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle=\sum_{j,m''}(-1)^{m''}\delta_{-m'}^{-m''}\mathcal{D}^{*(j)}_{m'',m}(R)$$

$$\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle=(-1)^{m'}\mathcal{D}^{*(j)}_{m',m}(R)$$

Therefore, using both,

$$\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle=(-1)^{m}\mathcal{D}^{(j)}_{-m',-m}(R)$$

$$\langle j,-m'|\Theta\mathcal{D}(R)|j,m\rangle=(-1)^{m'}\mathcal{D}^{*(j)}_{m',m}(R)$$

we have

$$(-1)^{m}\mathcal{D}^{(j)}_{-m',-m}(R)=(-1)^{m'}\mathcal{D}^{*(j)}_{m',m}(R)$$

and have proved

$$\mathcal{D}^{(j)*}_{m'm}(R)=(-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R)$$

c. This is equation 4.4.79 in Sakurai, and he explains that it works for either half integer $j$ or integer $j$. For half-integer $j$ we would set $\eta=+i$ in $\Theta=\eta e^{-i\pi J_y/\hbar}K$. This makes sense because

$$\mathcal{D}(R)\Theta|j,m\rangle=\mathcal{D}(R)(-1)^m|j,-m\rangle$$

with $i^2=-1$

$$\mathcal{D}(R)\Theta|j,m\rangle=\mathcal{D}(R)i^{2m}|j,-m\rangle$$

thus

$$\Theta|j,m\rangle=i^{2m}|j,-m\rangle$$

Problem 4.12
The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2)$$

Solve this problem exactly to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?

Solution:

Our first task at hand is to find the matrices for a spin 1 system, so that we can construct the Hamiltonian in matrix form. The eigenspace is 3 dimensional and spanned by $m_z = -1, 0, 1$. Defining the states

$$X_{+1}^z \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad X_0^z \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad X_{-1}^z \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and using

$$S_z X_{m_z}^z = m_z \hbar X_{m_z'}^z \qquad X_{m_z}^{z\dagger} X_{m_z'}^z = \begin{cases} 1 & m_z = m_z' \\ 0 & m_z \neq m_z' \end{cases}$$

$S_z$ is constructed like so:

$$S_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

I will use

$$S_x = \frac{1}{2}(S_+ + S_-) \qquad S_y = \frac{1}{2i}(S_+ - S_-)$$

to find $S_+$ and $S_-$. I remember the general form of them, but I need the formulas

$$S_\pm X_{m_z}^z = \sqrt{s(s+1) - m_z(m_z \pm 1)}\hbar X_{m_z \pm 1}^z$$

to find the coefficients in front. From $S_+ X_0^z = \sqrt{2}\hbar X_{+1}^z$ I see the coefficient is $\sqrt{2}\hbar$. So I can construct

$$S_+ = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_- = \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

So I have

$$S_x = \frac{\sqrt{2}\hbar}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \qquad S_y = \frac{\sqrt{2}\hbar}{2} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}.$$

Now I'm finally in a position to just write down the Hamiltonian in matrix form. Using $S_z, S_x, S_y$ I square them, and plug them into $H$.

$$S_x^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \qquad S_y^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

$$AS_z^2 = \hbar^2 \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}.$$

The Hamiltonian is

$$H = \hbar^2 \begin{bmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{bmatrix}.$$

Now it's just a eigenvalue problem.

$$0 = \begin{vmatrix} A-\lambda & 0 & B \\ 0 & -\lambda & 0 \\ B & 0 & A-\lambda \end{vmatrix}.$$

$$-\lambda(A-\lambda)^2 + \lambda B^2 = 0$$

The energy eigenvalues are

$$E_n = 0, \;\; \hbar^2(A \pm B)$$

The eigenvectors can be found like so:
With $\lambda = A + B$

$$Aa + Bc = (A+B)a \rightarrow a = c$$

$$0 = (A+B)b \rightarrow b = 0$$

With $\lambda = A - B$

$$Aa + Bc = (A-B)a \rightarrow c = -a$$

$$0 = (A-B)b \rightarrow b = 0$$

And lastly, orthogonality imposes for $\lambda = 0$

$$b = 1 \quad a = c = 0$$

So we have

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

or, expressed in $|S, S_z\rangle$,

$$e_1 = \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) \quad e_2 = \frac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle) \quad e_3 = |1,0\rangle$$

The Hamiltonian is invariant under time reversal. Using $\Theta \boldsymbol{S} \Theta^{-1} = -\boldsymbol{S}$ we see that indeed:

$$\begin{aligned}
\Theta H \Theta^{-1} &= \Theta A S_z^2 \Theta^{-1} + \Theta B S_x^2 \Theta^{-1} - \Theta B S_y^2 \Theta^{-1} \\
&= \Theta A S_z \Theta^{-1} \Theta S_z \Theta^{-1} + \Theta B S_x \Theta^{-1} \Theta S_x \Theta^{-1} - \Theta B S_y \Theta^{-1} \Theta S_y \Theta^{-1} \\
&= A(-S_z)(-S_z) + B(-S_x)(-S_x) - B(-S_y)(-S_y) \\
&= A S_z^2 + B(S_x^2 - S_y^2) \\
&= H
\end{aligned}$$

The eigenvalues transform under time reversal accordingly due to the formula

$$\Theta|l, m\rangle = (-1)^m |l, -m\rangle$$

So,

$$\Theta|e_1\rangle = \frac{1}{\sqrt{2}}(-|1,-1\rangle - |1,1\rangle) = -|e_1\rangle$$

$$\Theta|e_2\rangle = \frac{1}{\sqrt{2}}(-|1,-1\rangle + |1,1\rangle) = |e_2\rangle$$

$$\Theta|e_3\rangle = |1,0\rangle = |e_3\rangle$$